differential calc week 1-8

integral calc week 9-12

1.1 Scalar Function

- function

$f: A \to B \qquad a \to f(a)$

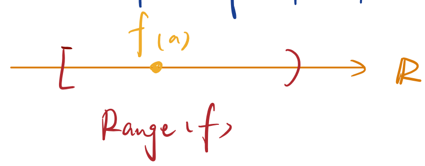
f ↑ function, A ↑ domain, B ↑ codomain

- def. scalar function

Let $f: A \to \mathbb{R}$ be a scalar function.

The range of f is the set $\mathrm{Range}(f) = \{ z \in \mathbb{R} : z \text{ is of the form } f(a) \text{ for some } a \in A \}$

$f: A \to \mathbb{R} \qquad$ domain : $A \subseteq \mathbb{R}^n$



eg. $f: \mathbb{R}^n \to \mathbb{R}. \qquad h(x_1, \cdots, x_n) = x_1 + \cdots + x_n$

$f(x, y) = e^x + e^y \qquad (0, \infty) \in \mathbb{R}.$ 只要是 subset of $\mathbb{R}$ 即可

convention : In M237, the domain of a "formula" will always the largest subset of $\mathbb{R}^n$. on which formula is well-defined.

Q. Let $f: \mathbb{R} \to \mathbb{R}$. $f(x, y) = x^2 + y^2$. What's Range(f) ?

Sol : $\mathrm{Range}(f) = \{ z \in \mathbb{R}. \ 0 \leq z < \infty \} = [0, \infty)$.

proof : $(\subseteq)$ Let $z \in \mathrm{Range}(f)$

$\therefore z = f(x, y)$ for some $(x, y) \in \mathbb{R}^2$

$= \underbrace{x^2}_{\geq 0} + \underbrace{y^2}_{\geq 0}$

$\therefore z \geq 0 \qquad z \in [0, \infty)$

$(\supseteq)$ Let $z \in [0, \infty) \qquad$ Then $z = f(\pm\sqrt{z}, 0)$

$\therefore z \in \mathrm{Range}(f)$

1.2 Geometric interpretation — $z = f(x, y)$

- level curves

(也称作 contour map / topographic map)
等高线图 / 地形图

$k = f(x, y)$
↑
range of f

domain : x, y

range : $f(x, y)$

DEFINITION
Level Curves

The **level curves** of a function $f(x, y)$ are the curves

$$f(x, y) = k$$

where k is a constant in the range of f.

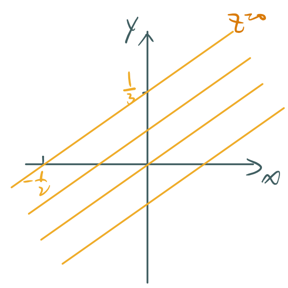
Q. Find level curves of function defined by $f(x,y) = 2x - 3y + 1$

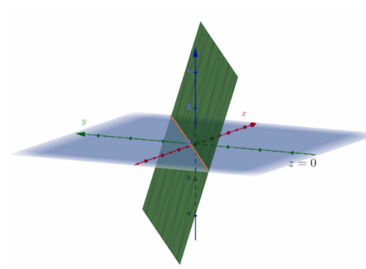
let $z = 2x - 3y + 1$

$z = 0 \quad 2x - 3y + 1 = 0$

$z = 1 \quad 2x - 3y + 1 = 1$

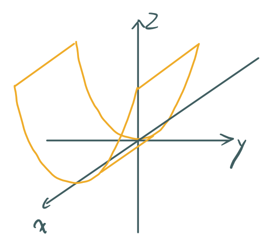
$z = 2 \quad 2x - 3y + 1 = 2$





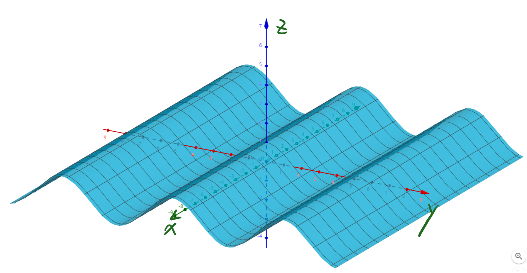
Q. Sketch $f(x,y) = y^2$

$z = y^2$



Q. Sketch $f(x,y) = \cos(x)$

$z = \cos x$



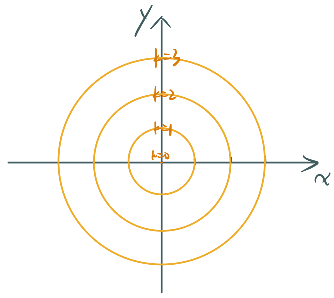
Q. Sketch the level curves of $f(x,y) = x^2+y^2$ and sketch surface $z = f(x,y)$

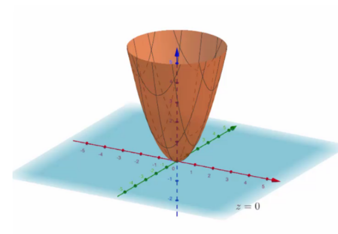
$R(f) = [0, \infty)$

$k=0 \quad . \quad x^2+y^2=0$

$k=1 \quad . \quad x^2+y^2=1$

$k=2 \quad . \quad x^2+y^2=2.$





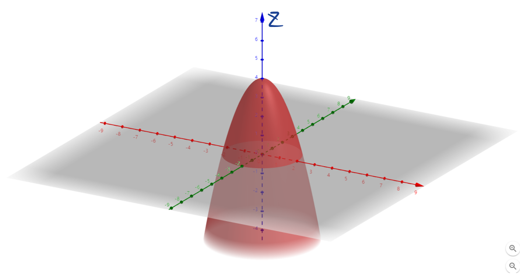
Q. Sketch $f(x,y) = 4-(x^2+y^2)$

let $z = 4-(x^2+y^2)$ range: $z \le 4$

$z=0 \quad x^2+y^2=4$

$z=1 \quad x^2+y^2=3$

$z=2 \quad x^2+y^2=2$



Q. Sketch $f(x,y) = \sqrt{4-(x^2+y^2)}$

let $z = \sqrt{4-(x^2+y^2)}$

domain: $4-(x^2+y^2) \ge 0 \quad x^2+y^2 \le 4$ range: $z \in [0,2]$

proof: $z \in Range(f)$

$\Leftrightarrow z = f(x,y)$

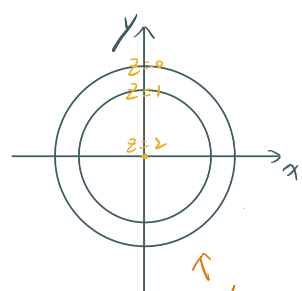
$\Leftrightarrow z = \sqrt{4-(x^2+y^2)} \quad 0 \le x^2+y^2 \le 4 \quad 2 \ge \sqrt{4-(x^2+y^2)}$

$\therefore z \ge 0. \quad z \le 2.$

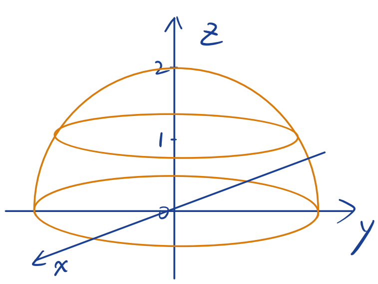
$z=0 \quad 0 = 4-(x^2+y^2) \quad x^2+y^2=4$

$z=1 \quad 1 = 4-(x^2+y^2) \quad x^2+y^2=3$

$z=2 \quad 4 = 4-(x^2+y^2) \quad x^2+y^2=0$



↑ contour plot

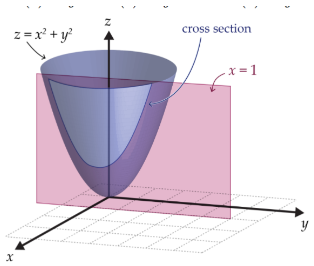


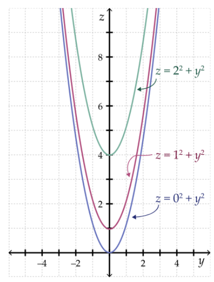
- cross section

DEFINITION
Cross Sections

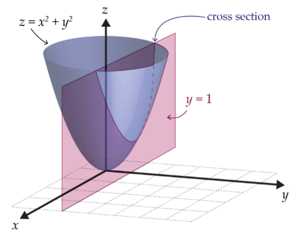
A **cross section** of a surface $z = f(x, y)$ is the intersection of $z = f(x, y)$ with a plane.

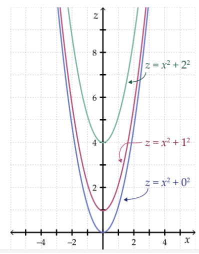
将 x 或 y 值定为 constant





The cross sections formed by intersecting $z = f(x, y)$ with $y = d$ for $d = 0, 1, 2$ are: $z = x^2 + (0)^2$, $z = x^2 + (1)^2$, and $z = x^2 + (2)^2$.

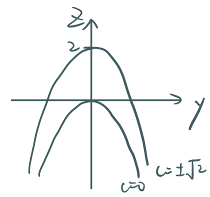


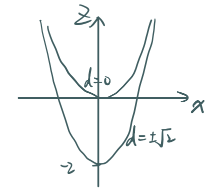


Q. Sketch the cross sections of $g(x,y) = x^2 - y^2$.

for $x = c$. $c = 0$ $z = -y^2$
$c = \pm\sqrt{2}$ $z = 2 - y^2$

for $y = d$. $d = 0$ $z = x^2$
$d = \pm\sqrt{2}$ $z = x^2 - 2$





- level surface

$f(x, y, z) = k \quad k \in R(f)$

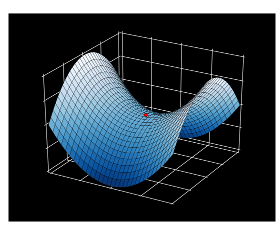
- level set

$\{\vec{x} \in \mathbb{R}^n \mid f(\vec{x}) = k\}$. for $k \in R(f)$

Function	General Form	Level Curves	Cross-Sections
Plane	$f(x,y) = ax + by + c$	Parallel lines	Parallel lines
Parabolic cylinder	$f(x,y) = ax^2$	Parallel lines	Horizontal lines or parabolas
Elliptic paraboloid	$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ where $a, b \in \mathbb{R}$ have the same sign	Circles or ellipses	Parabolas
Hyperbolic paraboloid	$f(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ where $a, b \in \mathbb{R}$ have different signs	Hyperbolas) (	Parabolas

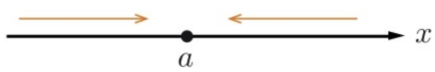
判断 level curves，需对 f 化简

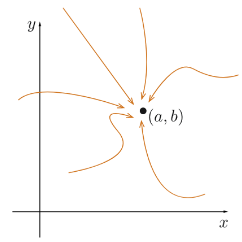
circles
parabolas
hyperbolas 双曲线
ellipse 椭圆
lines
paraboloid
saddle surface

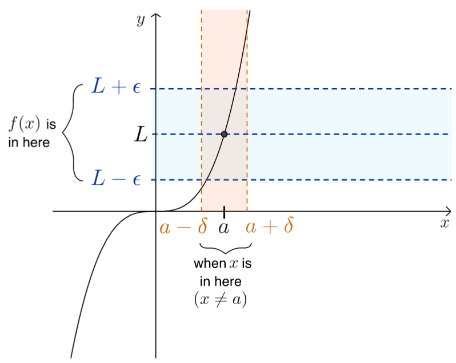


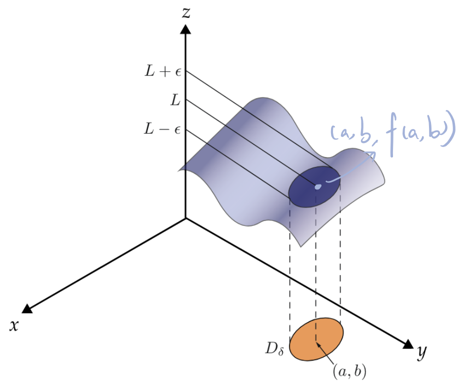
2.1 Definition of a limit.

- $\lim f(x)$. $\qquad\qquad$ $\lim f(x,y)$









$\forall \varepsilon > 0$. $\exists \delta > 0$. s.t.
if $0 < |x-a| < \delta$.
then $|f(x) - L| < \varepsilon$.

$\forall \varepsilon > 0$. $\exists \delta > 0$. s.t.
if $0 < \|(x,y) - (a,b)\| < \delta$.
then $|f(x,y) - L| < \varepsilon$

$\lim\limits_{x \to a} f(x) \Leftrightarrow \lim\limits_{x \to a^-} f(x) = L = \lim\limits_{x \to a^+} f(x)$

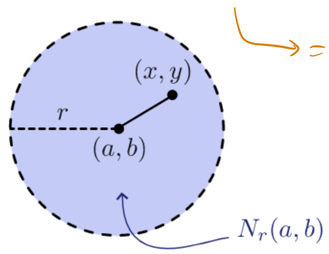
$\lim\limits_{(x,y) \to (a,b)} f(x,y) = L$

- def. r-neighbourhood

An ***r*-neighbourhood** of a point $(a,b) \in \mathbb{R}^2$ is a set

$$N_r(a,b) = \{(x,y) \in \mathbb{R}^2 \mid \underline{\|(x,y) - (a,b)\|} < r\}, \quad r \in \mathbb{R}$$

$= \sqrt{(x-a)^2 + (y-b)^2}$



Note that if $r < 0$, N_r is the empty set.

2.2 Limit Theorem

THEOREM 1 If $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ both exist, then

(a) $\lim_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$.

(b) $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right]$.

(c) $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$, provided $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$.

proof:

Let $\varepsilon > 0$. $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$. $\lim_{(x,y)\to(a,b)} g(x,y) = L_2$.

(a). By def of limit. $\exists\, \delta > 0$. s.t

if $0 \le \|(x,y)-(a,b)\| < \delta$. then $|f(x,y)-L_1| < \frac{1}{2}\varepsilon$ $|g(x,y)-L_2| < \frac{1}{2}\varepsilon$.

$$
\begin{aligned}
&|f(x,y)+g(x,y)-(L_1+L_2)| \\
&= |(f(x,y)-L_1)+(g(x,y)-L_2)| \\
&\le |f(x,y)-L_1| + |g(x,y)-L_2| \quad \text{by triangle inequality.} \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

(b) Let $\varepsilon > 0$. $\exists\, \delta_1 > 0$. $\delta_2 > 0$. s.t.

$\|(x,y)-(a,b)\| < \delta$. $\to$ then $|(f(x)-L_1)-0| < \sqrt{\varepsilon}$
$\to$ then $|(f(x)-L_2)-0| < \sqrt{\varepsilon}$.

$|f(x,y)-L_1||g(x,y)-L_2| < \varepsilon$.

$$
\begin{aligned}
&\lim_{(x,y)\to(a,b)} |f(x,y)\, g(x,y) + L_1L_2 - L_1 g(x,y) - L_2 \cdot f(x,y)| \\
&= \lim_{(x,y)\to(a,b)} |f(x,y)\, g(x,y) + L_1L_2 - L_1L_2 - L_2L_1| \\
&< \varepsilon.
\end{aligned}
$$

Q **Example 1**

Find the limit $\lim_{(x,y)\to(1,1)} \frac{x^2 - xy + y^2}{x^2 + y^2}$.

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - xy + y^2}{x^2+y^2} = \frac{\lim_{(x,y)\to(1,1)} x^2 - xy + y^2}{\lim_{(x,y)\to(1,1)} x^2 + y^2} \quad \text{(limit theorem 1)}$$

$$= \frac{\lim_{x\to1} x^2 - (\lim_{x\to1} x)(\lim_{y\to1} y) + \lim_{y\to1} y^2}{2}$$

$$= \frac{1}{2}$$

THEOREM 2 If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then the limit is unique.

proof: prove by contradiction: limit is not unique.

Let $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$. $\lim_{(x,y)\to(a,b)} f(x,y) = L_2$.

Assume $L_1 \neq L_2$.

$$|L_1 - L_2|$$
$$= \left|\lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} f(x,y)\right|$$
$$= \left|\lim_{(x,y)\to(a,b)} f(x,y) - f(x,y)\right|$$

$\therefore L_1 = L_2$ contradicts to $L_1 \neq L_2$

So, limit is unique.

2.3 Prove limit DNE.

证明 limit DNE：

approach limit along
$$\begin{cases} y=mx & \lim\limits_{(x,y)\to(0,0)} f(x,mx) \\ y=0 & \lim\limits_{(x,y)\to(0,0)} f(x,0). \\ x=0 & \lim\limits_{(x,y)\to(0,0)} f(0,y) \\ y=x & \lim\limits_{(x,y)\to(0,0)} f(x,x). \end{cases}$$

若其中两个值不等，则 limit DNE.

遇到 "$|x-1|$"

$\lim\limits_{x\to1^+} \neq \lim\limits_{x\to1^-}$

Example 1

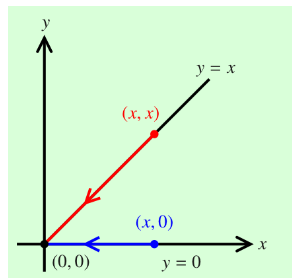
Let f be defined by $f(x,y)=\dfrac{xy}{x^2+y^2}$, for $(x,y)\neq(0,0)$. Prove that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

→ approach limit along $y=0$.

$$\lim_{(x,y)\to(0,0)} f(x,0) = \lim_{x\to0} \frac{x(0)}{x^2+0^2} = \lim_{x\to0} \frac{0}{x^2} = 0$$

→ approach limit along $y=x$.

$$\lim_{(x,y)\to(0,0)} f(x,x) = \lim_{x\to0} \frac{x^2}{x^2+x^2} = \frac{1}{2}.$$



→ ∵ $f(x,y)$ approach different values as (x,y) tends to $(0,0)$ along different path

∴ limit DNE

Example 2

Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does not exist.

→ approach limit along $y=mx$.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x(mx))}{x^2+(mx)^2} = \lim_{x\to0} \frac{\sin(mx^2)}{x^2(1+m^2)}$$
$$= \lim_{x\to0} \frac{2mx\cos(mx^2)}{2x(1+m^2)} \quad (\text{L'HR})$$
$$= \lim_{x\to0} \frac{m\cos(mx^2)}{1+m^2}$$

→ ∵ limit depends on m.

we get diffent limits along different lines $y=mx$.

∴ limit DNE

2.4 Prove Limit exist → Squeeze theorem

THEOREM 1 **(Squeeze Theorem)**

If there exists a function $B(x, y)$ such that

$$|f(x, y) - L| \le B(x, y), \quad \text{for all } (x, y) \ne (a, b)$$

in some neighborhood of (a, b) and $\lim_{(x,y)\to(a,b)} B(x, y) = 0$, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

proof:

Let $\varepsilon > 0$.

$\because \lim_{(x,y)\to(a,b)} B(x,y) = 0.$

$\therefore$ by def. of limit. there exist $\delta > 0$. s.t.

If $0 < \|(x,y)-(a,b)\| < \delta$, then $|f(x,y)-L| \le B(x,y) = |B(x,y)| < \varepsilon$.

$\because \forall (x,y) \ne (a,b) \quad B(x,y) \ge 0$

$\therefore$ by def. of limit. $\lim_{(x,y)\to(a,b)} f(x,y) = L$

Prove that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$

$f(x,y) = \frac{x^2 y}{x^2+y^2} \qquad L = 0$

→ $\because (x,y) \ne (0,0)$

$\therefore |f(x,y) - L| = \left| \frac{x^2 y}{x^2+y^2} - 0 \right| = \frac{x^2|y|}{x^2+y^2}$

→ $\because y^2 \ge 0 \quad \therefore x^2 \le x^2+y^2$

$\frac{x^2|y|}{x^2+y^2} \le \frac{(x^2+y^2)|y|}{x^2+y^2} = |y|$

$\therefore \forall (x,y) \ne (0,0) .\ 0 \le |f(x,y) - L| \le |y|$

→ take $B(x,y) = |y|$. $\lim_{(x,y)\to(0,0)} |y| = 0$

$\therefore$ by squeeze theorem, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$

Example 2

Prove that

$$\frac{|2x^2 - y^2|}{|x| + |y|} \le 2|x| + |y|, \quad \text{for all } (x, y) \neq (0, 0)$$

$$\begin{aligned}
|2x^2 - y^2| &= |2x^2 + (-y^2)| \\
&\le |2x^2| + |-y^2| \quad \text{by triangle inequality} \\
&= 2|x|^2 + |y|^2 \\
&\le 2|x|(|x| + |y|) + |y|(|x| + |y|) \quad (|x| \le |x| + |y| \quad |y| \le |x| + |y|) \\
&= (2|x| + |y|)(|x| + |y|)
\end{aligned}$$

$$\therefore \frac{|2x^2 - y^2|}{|x| + |y|} \le \frac{(2|x| + |y|)(|x| + |y|)}{|x| + |y|} = 2|x| + |y|$$

☆ determine whether $\lim_{(x,y)\to(a,b)} f(x,y)$ exist.

$f(x,y)$ cont at (a,b)?

- Y → limit exist
- N → 用 $x=a$. $y=b$. $y=f(x)$ 去 approach (可用 L'HR)
  取 $y=f(x)$ 尽量消掉 x.
  判断 $f(x,y)$ 是否 match.
  - Y → 用 $y=b+m(x-a)^k$ approach.
    判断 $f(x,y)$ 是否 match
    - Y → Squeeze theorem
      - Y → limit exist
      - N → 无法得出结论
    - N → limit DNE
  - N → limit DNE

Determine whether $\lim_{(x,y)\to(0,0)} \dfrac{x^2-|x|-|y|}{|x|+|y|}$ exists, and if so find its value.

→ approach limit along $y=mx$

$$\lim_{x\to 0} \frac{x^2-|x|-|m||x|}{|x|+|m||x|} = \lim_{x\to 0} \frac{|x|-(1+|m|)}{1+|m|} = -1$$

求 limit

∴ value along each line is $L=-1$

→ Prove $L=-1$. by Squeeze theorem.

证明 limit

$$\left|\frac{x^2-|x|-|y|}{|x|+|y|}-(-1)\right| = \left|\frac{x^2-|x|-|y|}{|x|+|y|}+\frac{|x|+|y|}{|x|+|y|}\right|$$
$$= \frac{x^2}{|x|+|y|}$$
$$= \frac{|x|\cdot|x|}{|x|+|y|}$$
$$\leq \frac{|x|(|x|+|y|)}{|x|+|y|} = |x| \quad (\text{Since } |x| \leq |x|+|y|)$$

Since $\lim_{(x,y)\to(0,0)} |x| = 0$ ∴ $\lim_{(x,y)\to(0,0)} \frac{x^2-|x|-|y|}{|x|+|y|} = -1$. by squeeze theorem.

☆ Determine whether $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$ exist.

→ approach limit along $y=0$. $y=mx$. we get $\lim = 0$

→ approach limit along $y=\sqrt{x}$. $(x\to 0^+)$

取 $y=f(x)$ 尽量消掉 x.

$$\lim_{x\to 0^+} \frac{x(\sqrt{x})^2}{x^2+(\sqrt{x})^4} = \lim_{x\to 0^+} \frac{x^2}{2x^2} = \frac{1}{2}$$

∵ $0 \neq \frac{1}{2}$

∴ limit DNE

2.5 Inequalities

- **Trichotomy Property:** For any real numbers a and b, one and only one of the following holds:
$$a = b, \qquad a < b, \qquad b < a$$
- **Transitivity Property:** If $a < b$ and $b < c$, then $a < c$.
- **Addition Property:** If $a < b$, then for all c, $a + c < b + c$.
- **Multiplication Property:** If $a < b$ and $c < 0$, then $bc < ac$.
- **Multiplicative Inverse Property:** If $ab > 0$ with $a < b$, then $\frac{1}{b} < \frac{1}{a}$. Note the change in order!

Prove that

$$\frac{|x^3 - y^3|}{x^2 + y^2} \le |x| + |y| \quad \text{for all} \quad (x, y) \neq (0, 0)$$

Does equality ever hold?

$|x^3 - y^3| \leq |x^3| + |y|^3$ by triangle inequality

$\leq (|x|+|y|)(x^2+y^2)$

$\frac{|x^3-y^3|}{x^2+y^2} \leq \frac{(|x|+|y|)(x^2+y^2)}{x^2+y^2} = |x|+|y|$ (as long as $(x,y) \neq (0,0)$)

$\therefore$ equality holds iff $(x,y) \neq (0,0)$

2. Let $f(x,y) = \dfrac{|x|^a|y|^b}{|x|^c+|y|^d}$ where a, b, c and d are positive numbers.

a. Prove that if $\frac{a}{c}+\frac{b}{d} > 1$ then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and equals zero.

b. Prove that if $\frac{a}{c}+\frac{b}{d} \le 1$, then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

b. → approach limit along $y=0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{0}{|x|^c} = 0.$$

→ approach limit along $y=x^r$ $(r \in \mathbb{R})$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{|x|^a|x^r|^b}{|x|^c+|x^r|^d}$$
$$= \lim_{x\to 0} \frac{|x|^{a+rb}}{|x|^c+|x|^{rd}}$$
$$= \lim_{x\to 0} \frac{|x|^c(|x|^{a+rb-c})}{|x|^c(1+|x|^{rd-c})}$$
$$= \lim_{x\to 0} \frac{|x|^{a+rb-c}}{1+|x|^{rd-c}}$$

✲ let $r = \frac{c-a}{b}$ $(b \neq 0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{|x|^{a-c+b\cdot\frac{c-a}{b}}}{1+|x|^{\frac{c-a}{b}d-c}}$$
$$= \lim_{x\to 0} \frac{1}{1+|x|^{\frac{d(c-a)}{b}-c}}$$

$\because \lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist iff $=0$

$\therefore \lim\limits_{x\to 0} |x|^{\frac{d(c-a)}{b}-c} \to \infty$

$\therefore \frac{d(c-a)}{b} - c < 0$ → $\lim\limits_{x\to 0} |0|^b = \infty$ $b<0$

$$\frac{dc-ad-bc}{b} < 0$$
$$dc-ad-bc<0$$
$$ad+bc > cd$$
$$\frac{ad+bc}{cd} > 1$$
$$\frac{a}{c}+\frac{b}{d} > 1$$

contradicts to $\frac{a}{c}+\frac{b}{d} \le 1$

3.1 Continuous function

- cont. function of one variable

f cont at $x=a \iff$ 1. $f(x)$ is defined.

2. $\lim_{x\to a} f(x)$ exists. $\lim_{x\to a^-} f(x)$ & $\lim_{x\to a^+} f(x)$ exists
$\lim_{x\to a^-} f = \lim_{x\to a^+} f$

3. $\lim_{x\to a} f(x) = f(a)$

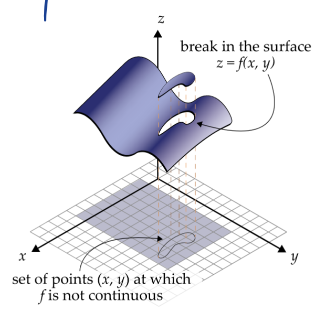
- cont function of two variable

def. $f(x,y)$ cont at $(a,b) \iff \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

f cont at $(a,b) \iff$ 1. $f(a,b)$ is defined.

2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$



Example 1

Let f be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Determine whether f is continuous at $(0,0)$.

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 = f(0,0)$

$\therefore f$ cont at $(0,0)$

Example 3

Consider f defined by

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2}, \quad \text{if } (x,y) \neq (0,0)$$

Is it possible for f to be defined at $(0,0)$ in such a way that the resulting function, whose domain is $\mathbb{R}^2$, is continuous at $(0,0)$?

$\because \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ DNE.

$\therefore f$ is not cont. at $(0,0)$

Example 2

Prove that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$.

$\hookrightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \neq 0$

$\to$ approach limit along $y=x$.

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$

$\to \because \lim \neq 0 \quad \therefore f$ is not cont at $(0,0)$

3.2 Continuity Theorem

DEFINITION
Operations on Functions

If $f(x, y)$ and $g(x, y)$ are scalar functions and $(x, y) \in D(f) \cap D(g)$, then:

1. the **sum** $f + g$ is defined by
$$(f + g)(x, y) = f(x, y) + g(x, y)$$
2. the **product** fg is defined by
$$(fg)(x, y) = f(x, y)g(x, y)$$
3. the **quotient** $\frac{f}{g}$ is defined by
$$\left(\frac{f}{g}\right)(x, y) = \frac{f(x, y)}{g(x, y)}, \quad \text{if } g(x, y) \neq 0$$

DEFINITION
Composite Function

For scalar functions $g(t)$ and $f(x, y)$ the **composite function** $g \circ f$ is defined by
$$(g \circ f)(x, y) = g(f(x, y))$$
for all $(x, y) \in D(f)$ for which $f(x, y) \in D(g)$.

Your Turn 2

Determine which compositions are allowed for $f(x,y) = \sin(x + y)$, $g(x,y) = \ln(x/y)$, $h(t) = \frac{1}{t}$ and $k(t) = |t|$ for an appropriately restricted domain. (Select all that apply)

- $(k \circ f)$ — Defined / **Not defined** ✗
- $(h \circ f)$ — Defined / **Not defined** ✗
- $(f \circ h)$ — Defined / **Not defined** ✓
- $(h \circ g)$ — **Defined** / Not defined ✓
- $(k \circ g)$ — **Defined** / Not defined ✓
- $(f \circ g)$ — Defined / **Not defined** ✓

$f \in [-1, 1]$
$g \in (0, \infty)$
$h \neq 0$
$k \in [0, \infty)$

Feedback

Hint: Check the domain and range for each of the functions.

$k(t)$: $t \in \mathbb{R}$. $f \in [-1, 1] \in \mathbb{R}$ $\therefore$ defined

$*$ $f(x,y)$ continuous on its domain $\rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

- Continuity theorem

THEOREM 1 If f and g are both continuous at (a, b), then $f + g$ and fg are continuous at (a, b).

proof: by def. continuous function. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

$\lim_{(x,y)\to(a,b)} g(x,y) = g(a,b)$

by def. sum & limit properties. $\lim_{(x,y)\to(a,b)} (f+g)(x,y)$

$= \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$

$= f(a,b) + g(a,b)$

$= (f+g)(a,b)$

by def. product & limit properties. $\lim_{(x,y)\to(a,b)} (fg)(x,y)$

$= \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y)$

$= f(a,b)\, g(a,b)$

$= (fg)(a,b)$

THEOREM 2 If f and g are both continuous at (a, b) and $g(a, b) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at (a, b).

THEOREM 3 If $f(x, y)$ is continuous at (a, b) and $g(t)$ is continuous at $f(a, b)$, then the composition $g \circ f$ is continuous at (a, b).

4.1 First Partial Derivatives

DEFINITION
Partial Derivatives

The **partial derivatives** of $f(x, y)$ are defined by

第一位为 variable →

$$D_1 f = \frac{\partial f}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$ → 令 y 为 constant

$$D_2 f = \frac{\partial f}{\partial y} = f_y = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$$ → 令 x 为 constant

provided that these limits exist.

→ 用于判断 f 在某一处是否 defined

→ 用于求 $f_x(a,b)$ 的值

For $f(x, y, z)$.

$$D_1 f = f_x = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

$$D_2 f = f_y = \lim_{h\to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

$$D_3 f = f_z = \lim_{h\to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

Example 1

Consider the function f defined by $f(x, y) = xe^{kxy}$ where k is a constant.

Determine $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 1e^{kxy} + x \cdot ky e^{kxy} = (1+kxy)e^{kxy}$$

$$\frac{\partial f}{\partial y} = x \cdot kx e^{kxy} = kx^2 e^{kxy}$$

Determine whether $\frac{\partial f}{\partial x}(0,0)$ exists for $f(x,y) = (x^3 + y^3)^{1/3}$.

→ find $\frac{\partial f}{\partial x}$ $\quad \frac{\partial f}{\partial x}(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$

→ determine whether $\frac{\partial f}{\partial x}$ exists at $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{(h^3 + 0^3)^{\frac{1}{3}} - 0}{h}$$

$$= 1$$

$\therefore \frac{\partial f}{\partial x}(0,0)$ exists, $= 1$.

4.2 Higher-order Partial Derivatives

DEFINITION
Hessian Matrix

The **Hessian matrix** of $f(x, y)$, denoted by $Hf(x, y)$, is defined as

$$Hf(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

- $\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_1^2 f := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$
- $\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_2 D_1 f := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ differentiate $\frac{\partial f}{\partial x}$ with respect to y.
- $\frac{\partial^2 f}{\partial y^2} = f_{yy} = D_2^2 f := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$
- $\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_1 D_2 f := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

Example 1

Let k be a constant. Find all the second partial derivatives of $f(x, y) = xe^{kxy}$.

$\frac{\partial f}{\partial x}(x,y) = e^{kxy} + kxye^{kxy}$

$\frac{\partial f}{\partial y}(x,y) = kx^2e^{kxy}$

- $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(e^{kxy} + kxye^{kxy}\right) = 2kye^{kxy} + k^2xy^2e^{kxy}$
- $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(e^{kxy} + kxye^{kxy}\right) = 2kxe^{kxy} + k^2x^2ye^{kxy}$
- $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(kx^2e^{kxy}\right) = 2kxe^{kxy} + k^2x^2ye^{kxy}$.
- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(kx^2e^{kxy}\right) = k^2x^3e^{kxy}$

THEOREM 1 **(Clairaut's Theorem)**

If f_{xy} and f_{yx} are defined in some neighborhood of (a, b) and are both continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

⊛ 如果有其中一个 not cont. 则 theorem 不成立. $f_{xy} \neq f_{yx}$

ep.

$$f_{xy} = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Q. $g(x, y, z, w) = xy^2\sin(w)z + yz^2e^{x+yw^2}$ 求 $g_{wzyzwzzxy}$

$g_z(x, y, z, w) = Az + Bz^2$

$g_{wzyzwzzxy} = g_{zzzz\cdots} = 0$

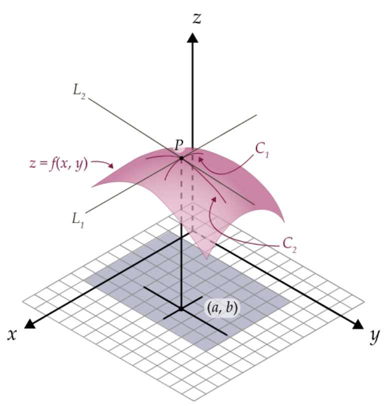
⊛ 根据 Clairaut's law, 如果 g cont. 则 里面字母可任意切换位置.

4.3 The tangent plane

DEFINITION
Tangent Plane

The **tangent plane** to $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$



C_1 : cross section $y=b$ of the surface

$z=f(x,b)$

$\frac{\partial f}{\partial x}(a,b)$ = slope of L_1 (tangent line of C_1)

C_2 : cross section $x=a$ of the surface

$z=f(a,y)$

$\frac{\partial f}{\partial x}(a,b)$ = slope of L_2 (tangent line of C_2)

Q. a. Find an equation for the tangent plane to the cone $f(x,y)=\sqrt{x^2+y^2}$ at the point $(3,-4,5)$.

The equation of your tangent plane should start with "z=". If it does not exist, write DNE in the input box.

$f(x,y)=\sqrt{x^2+y^2}$ $\quad P(3,-4,5)$

$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = x(x^2+y^2)^{-\frac{1}{2}} = \frac{3}{5}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y = y(x^2+y^2)^{-\frac{1}{2}} = -\frac{4}{5}$

$z = 5 + \frac{3}{5}(x-3) - \frac{4}{5}(y+4)$

b. Does the tangent plane you found in part a) pass through the origin?

- ✓ Yes
- ○ No

$5 - \frac{9}{5} - \frac{16}{5} = 0$

c. If you pick **any** point on the surface $f(x,y)=\sqrt{x^2+y^2}$, the tangent plane at that point will pass through the origin.

- ✓ True
- ○ False

$f_x(a,b) = \frac{a}{\sqrt{a^2+b^2}}$ $\qquad f_y(a,b) = \frac{b}{\sqrt{a^2+b^2}}$

$z = \sqrt{a^2+b^2} + \frac{a}{\sqrt{a^2+b^2}}(x-a) + \frac{b}{\sqrt{a^2+b^2}}(y-b)$

代入 $x=0$. $y=0$. 可得 $z=0$

∴ true

4.4 Linear Approximation for $z = f(x, y)$

- linear approximation of 1-D case

$y = f(x)$ 在 $(a, f(a))$ 点处 in tangent line : $y = f(a) + f'(a)(x-a)$

linearization of f at a : $L_a(x) = f(a) + f'(a)(x-a)$

当 $x \to a$, $L_a(x) \to f(x)$

- linear approximation of 2-D case

DEFINITION

Linearization

Linear Approximation

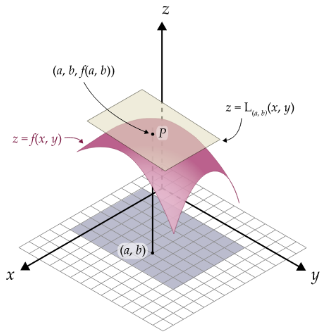
For a function $f(x, y)$ we define the **linearization** $L_{(a,b)}(x, y)$ of f at (a, b) by

$$L_{(a,b)}(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

We call the approximation

$$f(x, y) \approx L_{(a,b)}(x, y)$$

the **linear approximation** of $f(x, y)$ at (a, b).



Q. Use the linear approximation to approximate $\sqrt{(0.95)^3 + (1.98)^3}$.

Let $f(x,y) = \sqrt{x^3 + y^3}$ $\quad (a, b) = (1, 2)$

$\frac{\partial f}{\partial x} = \frac{3x^2}{2\sqrt{x^3 + y^3}}$ $\quad \frac{\partial f}{\partial y} = \frac{3y^2}{2\sqrt{x^3 + y^3}}$

$f(x, y) \approx L_{(1,2)}(x, y)$

$= f(1, 2) + f_x(1, 2)(x-1) + f_y(1, 2)(y-2)$

$= 3 + \frac{1}{2}(x-1) + 2(y-2)$

$\therefore \sqrt{(0.95)^3 + (1.98)^3} = f(0.95, 1.98) \approx 3 + \frac{1}{2}(-0.05) + 2 \cdot (-0.02) = 2.935$

Q.

Your Turn 1

Approximate $\sqrt{\sin\left(\frac{1}{10}\right) + \tan\left(\frac{3}{4}\right)}$ to 3 decimal places.

Use the approximate value 3.14 for π.

Let $f(x,y) = \sqrt{\sin x + \tan y}$ $\quad (a,b) = (0, \frac{\pi}{4})$

$\frac{\partial f}{\partial x} = \frac{1}{2}(\sin x + \tan y)^{-\frac{1}{2}} \cdot \cos x \quad \frac{\partial f}{\partial y} = \frac{1}{2}(\sin x + \tan y)^{-\frac{1}{2}} \cdot \sec^2 y$

$$f(x,y) \approx L_{(0,\frac{\pi}{4})}(x,y)$$
$$= f(0,\tfrac{\pi}{4}) + f_x(0,\tfrac{\pi}{4})x + f_y(0,\tfrac{\pi}{4})(y-\tfrac{\pi}{4})$$
$$= 1 + \tfrac{1}{2}x + (y - \tfrac{\pi}{4})$$
$$= 1.015$$

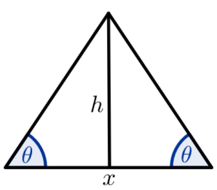
- Increment form of linear approximation

$$\underbrace{\Delta f}_{f(x,y)-f(a,b)} \approx \frac{\partial f}{\partial x}(a,b)\underbrace{\Delta x}_{x-a} + \frac{\partial f}{\partial y}(a,b)\underbrace{\Delta y}_{y-b}.$$

Example 2

An isosceles triangle has base 4 m, and equal angles of $\pi/4$. If the base is increased by 16 cm, and the equal angles are decreased by 0.1 radians, estimate the change in the area of the triangle.

Solution:
Let x be the length of the base of an isosceles triangle, θ be the measure of the equal angles, and h be the height of the triangle.



Then the area function can be written as

$$f(x,\theta) = \frac{1}{2}xh = \frac{1}{2}x\left(\frac{x}{2}\tan\theta\right) = \frac{1}{4}x^2\tan\theta$$

Recall that

$$\Delta f \approx \frac{\partial f}{\partial x}(a,b)(\Delta x) + \frac{\partial f}{\partial y}(a,b)(\Delta y)$$

Note that the change in x is $\Delta x = 16\text{ cm} = 0.16\text{ m}$ and the change in θ is $\Delta\theta = -0.1$ radians.
Also, $f_x = \frac{1}{2}x\tan\theta$ and $f_\theta = \frac{1}{4}x^2\sec^2\theta$. So we get $f_x(4,\pi/4) = 2$ and $f_\theta(4,\pi/4) = 8$.

Using the increment form of the linear approximation, we have

$$\Delta f \approx \frac{\partial f}{\partial x}(a,b)(\Delta x) + \frac{\partial f}{\partial y}(a,b)(\Delta y) = 2(0.16) - 8(0.1) = -0.48$$

Therefore, the area decreases approximately by 0.48 m^2.

4.5 Linear Approximation in higher dimensions

- linear approximation in $\mathbb{R}^3$.

$f(x, y, z)$

linearization of f at $\underline{a} = (a, b, c)$:

$$L_{\underline{a}}(x, y, z) = f(a) + f_x(a)(x-a) + f_y(a)(y-b) + f_z(a)(z-c)$$

$\because (x-a, y-b, z-c) \cdot (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a})) = f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + f_z(\vec{a})(z-c)$

$\therefore \quad (x-a, y-b, z-c) = (x, y, z) - (a, b, c)$

$\qquad \nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$

DEFINITION
Gradient

Suppose that $f(x, y, z)$ has partial derivatives at $\vec{\mathbf{a}} \in \mathbb{R}^3$. The **gradient** of f at $\vec{\mathbf{a}}$ is defined by

$$\nabla f(\vec{\mathbf{a}}) = \left(f_x(\vec{\mathbf{a}}), f_y(\vec{\mathbf{a}}), f_z(\vec{\mathbf{a}})\right)$$

DEFINITION
Linearization
Linear Approximation

Suppose that $f(\vec{\mathbf{x}})$, $\vec{\mathbf{x}} \in \mathbb{R}^3$, has partial derivatives at $\vec{\mathbf{a}} \in \mathbb{R}^3$. The **linearization** of f at $\vec{\mathbf{a}}$ is defined by

$$L_{\vec{\mathbf{a}}}(\vec{\mathbf{x}}) = f(\vec{\mathbf{a}}) + \nabla f(\vec{\mathbf{a}}) \cdot (\vec{\mathbf{x}} - \vec{\mathbf{a}}) \tag{4.5}$$

The **linear approximation** of f at $\mathbf{a}$ is

$$f(\vec{\mathbf{x}}) \approx f(\vec{\mathbf{a}}) + \nabla f(\vec{\mathbf{a}}) \cdot (\vec{\mathbf{x}} - \vec{\mathbf{a}}) \tag{4.6}$$

$\nabla f(a,b)(x-a, y-b)$
$= (f_x(a,b), f_y(a,b))(x-a, y-b)$
$= f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Example 1

Consider the function f defined by

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Find the gradient of f and the linear approximation for f at $\vec{a} = (1, 2, -2)$.

$\to \nabla f(x, y, z) = \left(\dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$\to$ evaluate $\nabla f(x, y, z)$ at $\vec{a} = (1, 2, -2)$, $\nabla f(\vec{a}) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$

$\to L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$

$\qquad = 3 + \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) \cdot (x-1, y-2, z+2)$

$\qquad = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) - \frac{2}{3}(z+2)$

$\therefore$ lin-app for f at $(1, 2, -2)$ is $f(x, y, z) \approx 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) - \frac{2}{3}(z+2)$

- linear approximation in $\mathbb{R}^n$

$\vec{a} \in \mathbb{R}^n$.

$\Delta \vec{x} = \vec{x} - \vec{a} = (x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n)$

$\nabla f(\vec{a}) = (D_1 f(\vec{a}), D_2 f(\vec{a}), \cdots, D_n f(\vec{a}))$

increment form of linear approximation for $f(\vec{x})$ is $\Delta f \approx \nabla f(\vec{a}) \cdot \Delta \vec{x}$

5.1 Definition of differentiability

– differentiability for functions

→ one variable

$R_{1,a}(x) = g(x) - L_a(x)$

DEFINITION **Taylor Remainder**

Assume that f is n times differentiable at $x = a$. Let

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

$R_{n,a}(x)$ is called the *n-th degree Taylor remainder function centered at $x = a$*.

THEOREM 1 If $g'(a)$ exists, then $\lim_{x\to a} \frac{|R_{1,a}(x)|}{|x-a|} = 0$ where

$$R_{1,a}(x) = g(x) - L_a(x) = g(x) - g(a) - g'(a)(x-a)$$

this means $R_{1,a}(x)$ is smaller than $r-a$,

if not, then $\frac{R_{1,a}(x)}{0} \neq 0$.

proof: $0 = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - f'(a)$

$= \lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{x-a}$

$= \lim_{x\to a} \frac{R_{1,a}(x)}{|x-a|} = 0$

the theorem says $R_{1,a}(x)$ tends to 0 faster than $|x-a|$.

符合上述 property 的是有 tangent line.

$\therefore \lim_{x\to a} \frac{|R_{1,a}(x)|}{|x-a|} = 0$ characterizes the tangent line at $(a, g(a))$

DEFINITION
Tangent Plane

Consider a function $f(x,y)$ which is differentiable at (a,b). The **tangent plane** of the surface $z = f(x,y)$ at $(a,b,f(a,b))$ is the graph of the linearization. That is, the tangent plane is given by

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

→ two variables

DEFINITION Differentiable

A function $f(x, y)$ is **differentiable** at (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x, y)|}{\|(x, y) - (a, b)\|} = 0$$

将(x,y)代入基于(a,b)所得到的target plane中

where

$$R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$$

THEOREM 2 If a function $f(x, y)$ satisfies

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - f(a, b) - c(x - a) - d(y - b)|}{\|(x, y) - (a, b)\|} = 0$$

then $c = f_x(a, b)$ and $d = f_y(a, b)$.

proof: $\because \lim_{(x,y)\to(a,b)} = 0$ $\therefore$ the limit is 0 along any path

→ approach through $y = b$.

$$0 = \lim_{x\to a} \frac{|f(x,b) - f(a,b) - c(x,a) - d(b-b)|}{\|(x,b) - (a,b)\|}$$

$$= \lim_{x\to a} \frac{|f(x,b) - f(a,b) - c(x,a)|}{|x-a|}$$

$$= \lim_{x\to a} \left| \frac{f(x,b) - f(a,b)}{x-a} - c \right|$$

$$= f_x(a,b) - c$$

$\therefore c = f_x(a,b)$

→ 同理 $d = f_y(a,b)$ QED

☆ Determine whether $f(x,y)$ is diff' at (a,b)

→ $f(x,y)$ is diff' at (a,b) if $\lim_{(x,y)\to(1,0)} \frac{|R_{1(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$

→ $L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$

$R_{(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$

→ 判断 $\lim_{(x,y)\to(1,0)} \frac{|R_{1(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} \overset{?}{=} 0$ < continuity theo / squeeze theo $= 0$; approach along ... $\neq 0$

- differentiable 的例子

Show that $f(x, y) = x^2 + y^2$ is differentiable at $(1, 0)$.

→ $f(x,y)$ is diff' at $(a,b) = (1,0)$ if $\lim_{(x,y)\to(1,0)} \frac{R_{(1,0)}(x,y)}{\|(x,y)-(1,0)\|} = 0$

→ 找 $R_{(1,0)}(x,y) = f(x,y) - L_{(1,0)}(x,y)$

$$L_{(1,0)}(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$
$$= 1 + 2(x-1)$$

- $f(1,0) = 1$
- $f_x(x,y) = 2x$ $\quad f_x(1,0) = 2$
- $f_y(x,y) = 2y$ $\quad f_y(1,0) = 0$

$$R_{(1,0)}(x,y) = f(x,y) - L_{(1,0)}(x,y)$$
$$= x^2 + y^2 - (1 + 2(x-1)) = (x-1)^2 + y^2$$

→ 证明 $\lim_{(x,y)\to(1,0)} \frac{R_{(1,0)}(x,y)}{\|(x,y)-(1,0)\|} = 0$ by continuity theorem

$$\lim_{(x,y)\to(1,0)} \frac{R_{(1,0)}(x,y)}{\|(x,y)-(1,0)\|} = \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 + y^2}{\sqrt{(x-1)^2+y^2}} = \lim_{(x,y)\to(1,0)} \sqrt{(x-1)^2+y^2} = 0$$

by continuity theorem

∴ The function is differentiable at $(1,0)$

- not differentiable 的例子

Determine whether $f(x, y) = \sqrt{|xy|}$ is differentiable at $(0, 0)$.

→ $f(x,y)$ is diff' at $(a,b) = (1,0)$ if $\lim_{(x,y)\to(1,0)} \frac{R_{(0,0)}(x,y)}{\|(x,y)-(0,0)\|} = 0$

→ $L_{(0,0)}(x,y) = f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y = 0$

$$R_{(0,0)}(x,y) = f(x,y) - L_{(0,0)}(x,y) = \sqrt{|xy|}$$

→ $$\lim_{(x,y)\to(0,0)} \frac{|R_{(0,0)}(x,y)|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{xy}}{\sqrt{x^2+y^2}}$$

approach limit along $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy}}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{\sqrt{|x|^2}}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}} \neq 0$$

∴ not differentiable

Q. Determine whether $f(x, y) = (x^2 + y^2)^{2/3}$ is differentiable at $(0, 0)$.

→ determine $L_{(0,0)}(x,y)$

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{(h^2)^{\frac{3}{2}}-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{(h^2)^{\frac{3}{2}}-0}{0} = 0$$

$$L_{(0,0)}(x,y) = f(x,y) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0$$

→ determine $\displaystyle\lim_{(x,y)\to(0,0)}\frac{|R_{(0,0)}|}{\|(x,y)-(0,0)\|}$

$$|R_{(0,0)}(x,y)| = f(x,y) - L_{(0,0)}(x,y) = |(x^2+y^2)^{\frac{2}{3}}|$$

$$\|(x,y)-(0,0)\| = \sqrt{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)}\frac{|R_{(0,0)}|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)}\frac{|(x^2+y^2)^{\frac{2}{3}}|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}(x^2+y^2)^{\frac{1}{6}} = 0$$

∴ diff' at $(0,0)$

→ determine whether $\frac{\partial f}{\partial x}$ cont at $(0,0)$

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)}\frac{2}{3}(x^2+y^2)^{-\frac{1}{3}}\cdot 2x = 0 = f_x(0,0)$$

∴ cont

→ determine whether $\frac{\partial f}{\partial y}$ cont at $(0,0)$

$$\lim_{(x,y)\to(0,0)} f_y(x,y) = \lim_{(x,y)\to(0,0)}\frac{2}{3}(x^2+y^2)^{-\frac{1}{3}}\cdot 2y = 0 = f_y(0,0)$$

∴ cont

∴ diff' at $(0,0)$

没有2种证明的方法

Instructors' Answer

Jennifer McKinnon
05/30/23

Theorem 2 is not about proving that \$f\$ is differentiable. It is showing a property of \$f_x\$ and \$f_y\$.

The definition of \$f\$ being differentiable involves the values \$f_x\$ and \$f_y\$. Theorem 2 is about showing that the choice to use \$f_x\$ and \$f_y\$ in the definition of differentiable was not simply convenient. Keep in mind that there are an infinite number of paths that we could in theory take derivatives along, so on that basis there isn't any reason to believe that \$f_x\$ and \$f_y\$ would be the only ones we could use in the definition of differentiability. But Theorem 2 shows that we were correct to use those values, as in fact they are the only values that would fit the pattern we wanted to use.

And so, while of some theoretical interest in the study of calculus, it isn't a theorem you are ever going to use. At least, not in this class.

Thanks

5.2 Differentiable $\Rightarrow$ Continuous

THEOREM 1 If $f(x, y)$ is differentiable at (a, b), then f is continuous at (a, b).

proof: Let error $R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$

$f(x,y) = f(a,b) + \nabla f(a,b)(x-a,\ y-b) + R_{1,(a,b)}(x,y)$ by def $L_{(a,b)}$

$\therefore R_{1,(a,b)}(x,y) = \dfrac{R_{1,(a,b)}(x,y)}{\|(x,y)-(a,b)\|}\ \|(x,y)-(a,b)\|$ $(x,y) \neq (a,b)$

By limit theorem,

$\because f$ diff'ble $\therefore \lim\limits_{(x,y)\to(a,b)} R_{1,(a,b)}(x,y) = 0$

$\therefore \lim\limits_{(x,y)\to(a,b)} f(x,y)$

$= \lim\limits_{(x,y)\to(a,b)} f(a,b) + \nabla f(a,b)(x-a,\ y-b) + R_{1,(a,b)}(x,y)$

for $\lim\limits_{(x,y)\to(a,b)} \nabla f(a,b)(x-a,\ y-b)$

$= \lim\limits_{(x,y)\to(a,b)} (f_x(a,b), f_y(a,b))(x-a,\ y-b)$

$= \lim\limits_{(x,y)\to(a,b)} \underbrace{f_x(a,b)}_{\text{constant}}\underbrace{(x-a)}_{0} + \underbrace{f_y(a,b)}_{\text{constant}}\underbrace{(y-b)}_{0}$

$= 0$

$\therefore \lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b) + 0 + 0 = f(a,b)$

$\therefore$ By def. f is cont at (a,b)

5.3 Continuous Partial Derivatives and differentiability

THEOREM 1 **(Mean Value Theorem)**

If $f(t)$ is continuous on the closed interval $[t_1, t_2]$ and f is differentiable on the open interval (t_1, t_2), then there exists $t_0 \in (t_1, t_2)$ such that

$$f(t_2) - f(t_1) = f'(t_0)(t_2 - t_1)$$

THEOREM 2 If $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous at (a, b), then $f(x, y)$ is differentiable at (a, b).

* $f(x,y)$ diff' at (a,b) $\not\Rightarrow$ $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ cont at (a,b)

* 若证明 $f(x,y)$ diff'，只需证 $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ cont. at (a,b)

EXERCISE 1 Discuss the validity of the approximation

$$\sqrt{1 + 3\tan x + \sin y} \approx 2 + \frac{3}{2}\left(x - \frac{\pi}{4}\right) + \frac{1}{4}y$$

the approximation is valid for $(x,y) \to (\frac{\pi}{4}, 0)$

→ We know that if the partial derivatives of f are continuous at (a, b), then f is differentiable and therefore

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x, y)|}{\|(x, y) - (a, b)\|} = 0$$

When this happens, we say that $L_{(a,b)}(x, y)$ is a good approximation of $f(x, y)$ near (a, b).

→ We are given the approximation $\sqrt{1 + 3\tan x + \sin y} \approx L_{(\frac{\pi}{4},0)} = 2 + \dfrac{3}{2}(x - \frac{\pi}{4}) + \dfrac{1}{4}(y - 0)$, which suggests that $f(x, y) = \sqrt{1 + 3\tan x + \sin y}$ and $(a, b) = (\frac{\pi}{4}, 0)$.

→ We need to check whether f_x and f_y are continuous at $(\frac{\pi}{4}, 0)$:

- $f_x = \dfrac{3\sec^2 x}{2\sqrt{1 + \sin(y) + 3\tan(x)}}$; using the continuity theorems, we determine that f_x is continuous at $(\frac{\pi}{4}, 0)$.

- $f_y = \dfrac{\cos(y)}{2\sqrt{1 + \sin(y) + 3\tan(x)}}$; using the continuity theorems, we determine that f_y is continuous at $(\frac{\pi}{4}, 0)$.

Since both f_x and f_y are continuous at $(\frac{\pi}{4}, 0)$, we can conclude that the approximation is valid for (x, y) close to $(\frac{\pi}{4}, 0)$.

$$
\begin{tikzcd}
 & f_x \ \& \ f_y \text{ cont. at } (a,b) \arrow[dl] \arrow[dr] \arrow[dd] & \\
f_x(a,b) \ \& \ f_y(a,b) \text{ exist} & & f \text{ is diff' at } (a,b) \arrow[ll] \arrow[dl] \\
 & f \text{ cont at } (a,b) &
\end{tikzcd}
$$

EXERCISE 2

For each of the following either give an example to prove that the statement is false, or justify why the statement is true.

(a) If f is not continuous at $(0,0)$, then f_x and f_y cannot both be continuous at $(0,0)$. T

(b) If f is continuous at $(0,0)$ and both $f_x(0,0)$ and $f_y(0,0)$ exist, then f is differentiable at $(0,0)$. F

(c) If f is not differentiable at $(0,0)$, then at least one of $f_x(0,0)$ or $f_y(0,0)$ does not exist. F

(d) If f is not differentiable at $(0,0)$, then f is not continuous at $(0,0)$. F

(e) If f is differentiable at $(0,0)$, then both $f_x(0,0)$ and $f_y(0,0)$ exist. T

(f) If f is differentiable at $(0,0)$, then f_x and f_y are both continuous at $(0,0)$. F

counter example for (b) (c) (d)

$$
f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}
$$

counter example for (f)

$$
f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}
$$

6.1 Chain Rule in 2 dimensions

- Chain Rule for $f(x(t))$

$$T = f(x(t))$$

$$\frac{dT}{dt} = \frac{dT}{dx} \cdot \frac{dx}{dt}$$

- Chain Rule for $f(x(t), y(t))$

THEOREM 1 **(Chain Rule)**

Let $G(t) = f(x(t), y(t))$, and let $a = x(t_0)$ and $b = y(t_0)$. If f is differentiable at (a, b) and $x'(t_0)$ and $y'(t_0)$ exist, then $G'(t_0)$ exists and is given by

$$G'(t_0) = f_x(a, b)x'(t_0) + f_y(a, b)y'(t_0)$$

proof:

by def. derivative $\quad G'(t_0) = \lim_{t \to t_0} \frac{G(t) - G(t_0)}{t - t_0}$

by def of $G(t)$ $\quad G(t) - G(t_0) = f(x(t), y(t)) - f(x(t_0), y(t_0))$

$\because$ f is differentiable

$\therefore f(x, y) = f_x(a, b)(x-a) + f_y(a, b)(y-b) + R_{1,(a,b)}(x, y)$ where $\frac{|R_{1,(a,b)}(x,y)|}{\sqrt{(x-a)^2+(y-b)^2}} \to 0$

$\because a = x(t_0),\ b = y(t_0)$

$\therefore \frac{G(t) - G(t_0)}{t - t_0} = f_x(a,b)\left(\frac{x(t) - x(t_0)}{t - t_0}\right) + f_y(a,b)\left(\frac{y(t) - y(t_0)}{t - t_0}\right) + \frac{R_{1,(a,b)}(x(t), y(t))}{t - t_0}$

$\therefore \lim_{t \to t_0} \frac{R_{1,(a,b)}(x(t), y(t))}{t - t_0} = 0$

Let $E(x, y) = \begin{cases} \frac{R_{1,(a,b)}(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} & (x,y) \neq (a,b) \\ 0 & (x,y) = (a,b) \end{cases}$

$\therefore R_{1,(a,b)}(x,y) = E(x,y)\sqrt{(x-a)^2+(y-b)^2} \qquad \forall (x,y)$

$\because a = x(t_0)\ b = y(t_0) \quad \therefore \frac{|R_{1,(a,b)}(x(t), y(t))|}{|t - t_0|} = E(x(t), y(t))\sqrt{\left(\frac{x(t) - x(t_0)}{t - t_0}\right)^2 - \left(\frac{y(t) - y(t_0)}{t - t_0}\right)^2}$

$\lim_{t \to t_0} \frac{|R_{1,(a,b)}(x(t), y(t))|}{|t - t_0|} = 0$. (Since $E(a,b) = 0$)

Example 1

Use the Chain Rule to find $\dfrac{df}{dt}$ for $f(x,y) = xy^3 - x^3y$ with $x(t) = t^2+1$ and $y(t) = t^2-1$ at $t_0 = 1$.

$\rightarrow$ find a & b

$a = x(1) = 1^2+1 = 2$ $\qquad$ $b = y(1) = 1^2 - 1 = 0$

$\rightarrow$ find $x'(t_0)$ & $y'(t_0)$

$x'(t) = 2t \qquad x'(1) = 2 \qquad y'(t) = 2t \qquad y'(1) = 2$

$\rightarrow G'(t_0)$

$\because f(x,y)$ is diff' at $(a,b) = (2,0)$. $x'(1)$ & $y'(1)$ exist. $\therefore G'(1)$ exist

$G'(t_0) = f_x(a,b)x'(t_0) + f_y(a,b)y'(t_0) = \underline{f_x(2,0)}x'(1) + \underline{f_y(2,0)}y'(1)$

$\rightarrow$ find $f_x(2,0)$ & $f_y(2,0)$

$f_x(x,y) = y^3 - 3x^2y \qquad f_x(2,0) = 0 \qquad f_y(x,y) = 3xy^2 - x^3 \qquad f_y(2,0) = -8.$

$\therefore G'(1) = 0\cdot 2 + (-8)\cdot 2 = -16.$

Example 2

Suppose that the temperature at position (x,y) in a lake is

$$T(x,y) = 10e^{-0.1(x^2+y^2)}$$

The path of a swimmer swimming on the lake is

$$x(t) = 2\cos t, \quad y(t) = 4\sin t$$

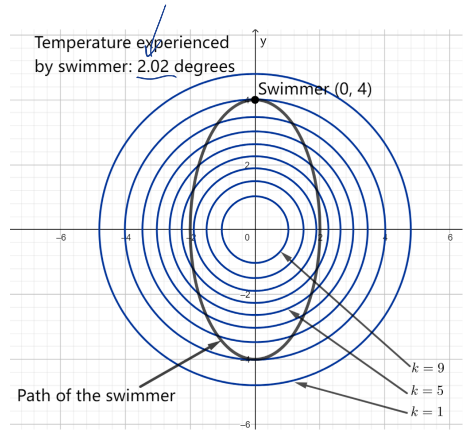
Find the rate of change of the lake's temperature as experienced by the swimmer at time $t = \pi/2$.

$T(t) = T(x(t), y(t)) \qquad \dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}.$ by Chain Rule

$\rightarrow \dfrac{dx}{dt}$ & $\dfrac{dy}{dt}$ at $t = \frac{\pi}{2}$

$\dfrac{dx}{dt} = 2\sin t = -2 \qquad \dfrac{dy}{dt} = 4\cos t = 0.$

$k = 10e^{-1.6} \approx 2.02$



$\rightarrow (x,y)$

$(2\cos(\frac{\pi}{2}), 4\sin(\frac{\pi}{2})) = (0,4)$

$\rightarrow \dfrac{\partial T}{\partial x}$ & $\dfrac{\partial T}{\partial y}$ at $(0,4)$

$\dfrac{\partial T}{\partial x} = -2xe^{-0.1(x^2+y^2)} = 0 \qquad \dfrac{\partial T}{\partial y} = -2ye^{-0.1(x^2+y^2)} = -8e^{-1.6}$

$\rightarrow \dfrac{dT}{dt}(\frac{\pi}{2}) = -2(0) + 0(-8e^{-1.6}) = 0$

Your Turn 1

Let

$$T(t) = \ln(1 + x^2 + y^2), \quad \text{with} \quad x(t) = e^t \sin t, \quad y(t) = 2e^t \cos t$$

Calculate $\dfrac{dT}{dt}$ when $t = 0$ in two different ways:

Method 1: Substitute x and y in T

$$T = \ln\left(1 + (e^t \sin t)^2 + (2e^t \cos t)^2\right)$$

$$\frac{dT}{dt} = \frac{2e^t \sin t \cdot (e^t \sin t + e^t \cos t) + 2 \cdot 2e^t \cos t\,(2e^t \cos t - 2e^t \sin t)}{1 + (e^t \sin t)^2 + (2e^t \cos t)^2}$$

$$= \frac{2e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + 8e^{2t} \cos^2 t - 8e^{2t} \cos t \sin t}{1 + (e^t \sin t)^2 + (2e^t \cos t)^2}$$

$$= \frac{8e^0}{1 + (2\cos 0)^2}$$

$$= \frac{8}{5}$$

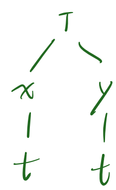
Method 2: Evaluate $\frac{dx}{dt}(0)$, $\frac{dy}{dt}(0)$, $\frac{\partial T}{\partial x}(0, 2)$ and $\frac{\partial T}{\partial y}(0, 2)$, and apply the Chain Rule.

$$\frac{dx}{dt}(0) = e^t \sin t + e^t \cos t = 1$$

$$\frac{dy}{dt}(0) = 2e^{2t} \cos^2 t - 2e^{2t} \cos t \sin t = 2$$

$$\frac{\partial T}{\partial x}(0, 2) = \frac{2x}{1 + x^2 + y^2} = 0$$

$$\frac{\partial T}{\partial y}(0, 2) = \frac{2y}{1 + x^2 + y^2} = \frac{4}{5}$$



$$\frac{dT}{dt}(0) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt}$$

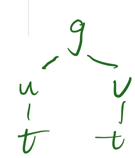
$$= 0 + \frac{4}{5} \cdot 2$$

$$= \frac{8}{5}$$

Your Turn 2

Define $f(t) = g(1 + t^2, 1 - t^2)$. If $\nabla g(2, 0) = (3, 4)$, find $f'(1)$. What condition on g will guarantee the validity of your work?

Let $g = g(u, v)$. So $f(t) = g(u(t), v(t))$



$u(t) = 1 + t^2 \quad \frac{du}{dt} = 2t \qquad v(t) = 1 - t^2 \quad \frac{dv}{dt} = -2t$

To apply Chain's Rule, we require g is diff' at $(u(1), v(1)) \to (2, 0)$

$$f'(1) = \frac{\partial g}{\partial u} \cdot \frac{du}{dt} + \frac{\partial g}{\partial v} \cdot \frac{dv}{dt} = 3 \cdot 2 + 4 \cdot (-2) = -2$$

- The vector form of basic chain rule.

We can use the dot product to rewrite the Chain Rule into a vector form. In particular, if we have

$$T(t) = f(x(t), y(t))$$

where $f(x, y)$, $x(t)$, and $y(t)$ are differentiable, then

$$\begin{aligned} \frac{dT}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\ &= \nabla f \cdot \frac{d\mathbf{x}}{dt} \end{aligned}$$

So, we have

$$\frac{d}{dt} f(\mathbf{x}(t)) = \nabla f(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt}(t)$$

with $\mathbf{x}(t) = (x(t), y(t))$.

Example 4

Let the temperature at position (x, y, z) in the vicinity of the planet Mercury be given by $T = T(x, y, z)$ where T is differentiable. If the path of a spaceship is $(x(t), y(t), z(t))$, then write the Chain Rule for $\frac{dT}{dt}$.

$$\frac{dT}{dt} = \nabla T(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))$$

$$= T_x(x(t), y(t), z(t)) \cdot x'(t) + T_y(x(t), y(t), z(t)) \cdot y'(t) + T_z(x(t), y(t), z(t)) \cdot z'(t)$$

Your Turn

A differentiable function $f(x, y, z)$ is given and $g(t)$ is defined by

$$g(t) = f(x, y, z)$$

where $x(t) = t$, $y(t) = t^2$, and $z(t) = t^3$.

Find $g'(1)$ if $\nabla f(1, 1, 1) = \left(2, \frac{1}{2}, 1\right)$.

$$g'(t) = \nabla f(t, t^2, t^3) \cdot (1, 2t, 3t^2)$$

$$g'(1) = \nabla f(1, 1, 1) \cdot (1, 2(1), 3(1)^2)$$

代入t.

代入∇f

$$= \left(2, \tfrac{1}{2}, 1\right) \cdot (1, 2, 3)$$

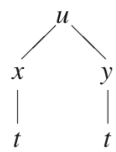
$$= 6$$

6.2 Extension of Basic Chain Rule

$$u = f(x, y), \quad \text{with} \quad x = x(t), \quad y = y(t)$$

In this situation, the different variables are referred to as follows:

u : dependent variable
x, y : intermediate variables
t : independent variable



$u(s,t)$. 将 t 视为 constant. 对 s 视为 variable 求导

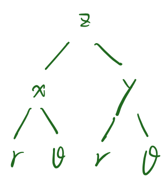
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$\frac{\partial u}{dt}$ = rate of chang wrt x + rate of change wrt y.

$$= \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

EXAMPLE 1 Let $z = f(x, y)$, where $x = r\cos\theta$, and $y = r\sin\theta$. Assuming that f is differentiable, verify that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)\right)^2$$

$$= \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta$$

$$= \left(\frac{\partial f}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial f}{\partial y}\right)^2(\cos^2\theta + \sin^2\theta)$$

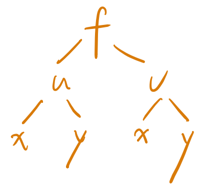
$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Your Turn 1

Let g be defined as in the example above, that is,

$$g(x,y) = f(\overbrace{2xy}^{u}, \overbrace{x^2 - y^2}^{v})$$

where f is differentiable with $\nabla f(2,0) = (2,3)$. Calculate $\frac{\partial g}{\partial y}(1,1)$.



$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}(u(x,y), v(x,y))\,\underline{\frac{\partial u}{\partial y}(x,y)} + \frac{\partial f}{\partial v}(u(x,y), v(x,y))\,\underline{\frac{\partial v}{\partial y}(x,y)}$$

$$= \frac{\partial f}{\partial u}(u(x,y), v(x,y))\,\underline{(2x)} + \frac{\partial f}{\partial v}(u(x,y), v(x,y))\,\underline{(-2y)}$$

$(x,y) = (1,1)$

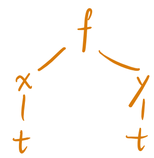
$$\frac{\partial g}{\partial y}(1,1) = 2 \cdot \frac{\partial f}{\partial u}(2,0) + 2\,\frac{\partial f}{\partial u}(2,0) = -2$$

Your Turn 2

A function g is defined by

$$g(t) = f(h(t)+t, h(t)-t)$$

where $f(x,y)$ and $h(t)$ are both differentiable. Write the Chain Rule for $g'(t)$.



$$g'(t) = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}\,\underline{(h'(t)+1)} + \frac{\partial f}{\partial y} \cdot \underline{(h'(t)-1)}$$

if $h(1) = 2$, $h'(1) = 3$ and $\nabla f(3,1) = (2,-3)$, find $g'(1)$.

$$g'(1) = \frac{\partial f}{\partial x}(h(1)+1, h(1)-1) \cdot (h'(t)+1) + \frac{\partial f}{\partial y}(h(1)+1, h(1)-1) \cdot (h'(t)-1)$$

$$= 2 \cdot 4 - 3 \cdot 2$$

$$= 2$$

Generalized Chain Rule

The Chain Rule can be generalized to functions having any number of independent variables in the following way:

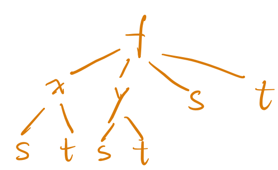
Let $w = f(x_1, \ldots, x_m)$ be a differentiable function of m independent variables and for $1 \le i \le m$ let $x_i = x_i(t_1, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for $1 \le j \le n$.

Your Turn 5

Let $u(s,t) = f(x(s,t), y(s,t), s, t)$. Write the Chain Rule for $\frac{\partial u}{\partial s}$, showing the functional dependence explicitly.



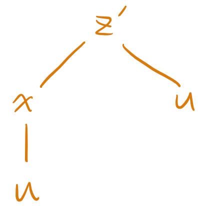
$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}$$

6.3 The Chain Rule for Second Partial Derivatives

Example 1

If $z = f(x)$ where f is twice differentiable and $x = e^u$, verify that

$$z''(u) = x^2 f''(x) + x f'(x)$$



$$z'(u) = f'(x) \cdot e^u$$

$$\begin{aligned} z''(u) &= \frac{\partial z'(u)}{\partial x}\frac{dx}{du} + \left(\frac{\partial z'(u)}{\partial u}\right)_x \\ &= (f''(x)e^u)e^u + f'(x)e^u \\ &= x^2 f''(x) + x f'(x) \end{aligned}$$

Remark

Observe, if we had substituted in $x = e^u$ at the beginning, we would get

$$z'(u) = f'(e^u)e^u$$

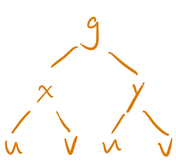
Hence, taking the derivative with respect to u we would get

$$\begin{aligned} z''(u) &= \frac{d}{du}(f'(e^u))e^u + f'(e^u)\frac{d}{du}(e^u) && \text{by the product rule} \\ &= \left(f''(e^u)\frac{d}{du}(e^u)\right)e^u + f'(e^u)e^u && \text{by the Chain Rule} \\ &= (f''(e^u)e^u)e^u + f'(e^u)e^u \end{aligned}$$

which matches with the result above. Thus, we see that our dependence diagram algorithm not only calculates the necessary Chain Rules, but also includes the necessary product rules.

Example 2

Let $g(u,v) = f(\underbrace{u^2 - v^2}_{x(u,v)}, \underbrace{2uv}_{y(u,v)})$. Express $(g_u)^2 + (g_v)^2$ and $g_{uu} + g_{vv}$ in terms of the partial derivatives of f. What hypothesis must f satisfy?



$$g_u = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2u f_x + 2v f_y$$

$$g_v = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2v f_x + 2u f_y$$

$$\begin{aligned} g_{uu} &= 2f_x + 2u(2u f_{xx} + 2v f_{xy}) + 0 + 2v(2u f_{yx} + 2v f_{yy}) \\ &= 2f_x + 4u^2 f_{xx} + 4uv f_{xy} + 4uv f_{yx} + 4v^2 f_{yy} \end{aligned}$$

$$\begin{aligned} g_{vv} &= -2f_x - 2v(-2v f_{xx} + 2u f_{xy}) + 0 + 2u(-2v f_{yx} + 2u f_{yy}) \\ &= -2f_x + 4v^2 f_{xx} - 4uv f_{xy} - 4uv f_{yx} + 4u^2 f_{yy} \end{aligned}$$

由此可得 $(g_u)^2 + (g_v)^2$ & $g_{uu} + g_{vv}$

Your Turn 1

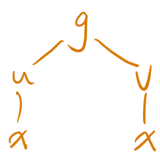
Let $g(u,v) \in C^2$ be a function, and let f be defined by

$$f(x) = g(x, 2x)$$

Find the values of a, b, c that would make the following statement true:

$$f''(x) = a g_{uu} + b g_{uv} + c g_{vv}$$

let $u = x$ $v = 2x$



$$f'(x) = g_u \cdot 1 + g_v \cdot 2$$

$$f''(x) = g_{uu} \cdot 1 + g_{uv} \cdot 2 + g_{vu} \cdot 2 + 2 g_{vv} \cdot 2$$

$$= g_{uu} + 4 g_{uv} + 4 g_{vv}$$

Your Turn 2

A twice-differentiable function $g(t)$ is given, and f is defined by

$$f(x,y) = g(xy)$$

Find the values of a, b that would make the following statement true:

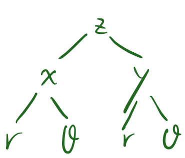
$$x^a f_{xx} = y^b f_{yy}$$

$f_x = y g'(xy)$ $\quad f_{xx} = y^2 g''(xy)$

$f_y = x g'(xy)$ $\quad f_{yy} = x^2 g''(xy)$

$\therefore x^2 f_{xx} = x^2 y^2 g''(xy) = y^2 f_{yy}$

Q Let $z = f(x,y)$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$ with $f \in C^2$. Calculate $\dfrac{\partial^2 z}{\partial\theta\partial r}$.



$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial f}{\partial x} \cdot (-r\sin\theta) + \frac{\partial f}{\partial y} (r\cos\theta)$$

$$\frac{\partial^2 z}{\partial\theta\partial r} = \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x} \cdot (-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)\right)$$

$$\frac{\partial}{\partial\theta}\left(\cos\theta \frac{\partial f}{\partial x}\right) = -\sin\theta \frac{\partial f}{\partial x} + \cos\theta \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right)$$

$$= -\sin\theta \frac{\partial f}{\partial x} + \cos\theta\left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial y}{\partial\theta}\right)$$

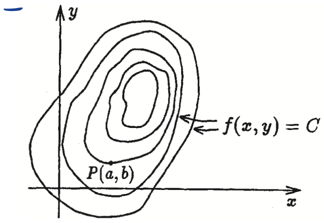
$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial y}\right) = \cos\theta \frac{\partial f}{\partial y} + \sin\theta \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)$$

$$= \cos\theta \frac{\partial f}{\partial y} + \sin\theta\left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial y}{\partial\theta}\right)$$

$$\frac{\partial^2 z}{\partial\theta\partial r} = -\sin\theta \frac{\partial f}{\partial x} + \underbrace{\cos\theta\left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial y}{\partial\theta}\right)}_{\frac{\partial f}{\partial y} - r\sin\theta \cdot \frac{\partial f}{\partial y}} + \cos\theta \frac{\partial f}{\partial y} + \underbrace{\sin\theta\left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial y}{\partial\theta}\right)}_{\sin\theta \frac{\partial^2 f}{\partial x \partial y} + r\cos\theta \frac{\partial^2 f}{\partial y^2}}$$

7.1 Directional Derivatives

-



level curve $f(x,y)=C$

目的: define a derivative

unit vector $\|\hat{u}\|=1$

f 关于点 (a,b) 在 $\hat{u}=(u_1,u_2)$ 方向上的 rate of change

L: line through (a,b) in direction $\hat{u}$: $(x,y)=(a,b)+s\hat{u} = (a+s\hat{u},\ b+s\hat{u})$

DEFINITION

Directional Derivative

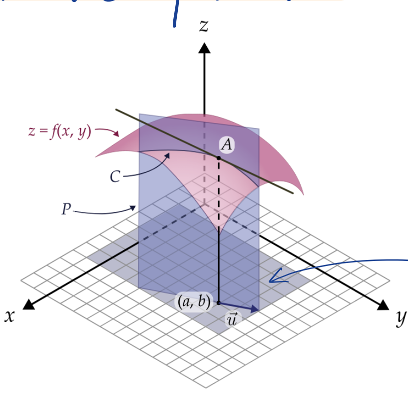
The **directional derivative** of $f(x,y)$ at a point (a,b) in the direction of a unit vector $\hat{u}=(u_1,u_2)$ is defined by

$$D_{\hat{u}}f(a,b) = \frac{d}{ds}f(a+su_1, b+su_2)\Big|_{s=0}$$

provided the derivative exists.

$$D_{\hat{u}}f(a,b) = \lim_{t\to 0}\frac{f((a,b)+t\vec{u})-f(a,b)}{t}$$

- Geometric representation



(x,y) 位置.

$z=F(x,y)$: 物理量

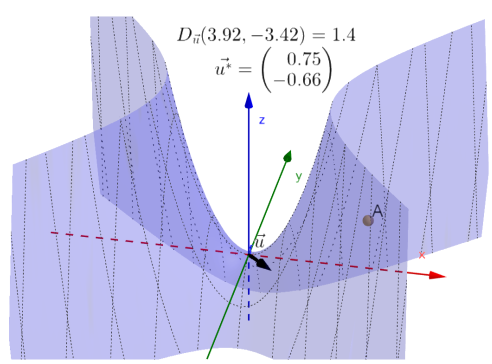
$D_{\hat{u}}f(a,b)$: rate of change

cross-section C at point A.

Example 1

Find the directional derivative of $f(x,y)=x^2-y^2$ at the point $(1,2)$ in the direction of the vector $\vec{u}=\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)$.

$$D_{\hat{u}}f(a,b) = \frac{d}{ds}f(a+su_1,\ b+su_2)\Big|_{s=0}$$
$$= \frac{d}{ds}f\left(1+\frac{1}{\sqrt{5}}s,\ 2+\frac{2}{\sqrt{5}}s\right)\Big|_{s=0}$$
$$= \frac{d}{ds}\left[\left(1+\frac{1}{\sqrt{5}}s\right)^2-\left(2+\frac{2}{\sqrt{5}}s\right)^2\right]\Big|_{s=0}$$
$$= 2\left(1+\frac{1}{\sqrt{5}}s\right)\cdot\frac{1}{\sqrt{5}} - 2\cdot\frac{2}{\sqrt{5}}\left(2+\frac{2}{\sqrt{5}}s\right)$$
$$= -\frac{6}{\sqrt{5}}$$



THEOREM 1

(1) $D_{\hat{i}} f = f_x$

(2) $D_{\hat{j}} f = f_y$

THEOREM 2 If $f(x, y)$ is differentiable at (a, b) and $\hat{u} = (u_1, u_2)$ is a unit vector, then

$$D_{\hat{u}} f(a, b) = \nabla f(a, b) \cdot \hat{u}$$

proof: apply Chain Rule.

$$D_{\hat{u}}(a,b) = \frac{d}{ds} f(a+su_1, b+su_2)\Big|_{s=0}$$

$$= \left[D_1 f(a+su_1, b+su_2)\frac{d}{ds}(a+su_1) + D_2 f(a+su_1, b+su_2)\frac{d}{ds}(b+su_2)\right]\Big|_{x=0}$$

$$= \left[D_1 f(a+su_1, b+su_2)\,u_1 + D_2 f(a+su_1, b+su_2)\,u_2\right]\Big|_{x=0}$$

$$= D_1 f(a,b)\,u_1 + D_2 f(a,b)\,u_2$$

$$= \nabla f(a,b) \cdot (u_1, u_2)$$

Example 2

Find the directional derivative of $f(x, y) = 2x^3 + 4xy^2 + y$ at the point $(-1, 1)$ in the direction of the vector $\vec{u} = (1, 1)$.

a b

$\nabla f(x,y) = (6x^2+4y^2,\ 8xy+1)$ $\quad\nabla f(-1,1) = (10, -7)$

$\vec{u^*} = \frac{(1,1)}{\|(1,1)\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D_{\vec{u^*}} f(-1,1) = (10,-7)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$

Your Turn 2

Find the directional derivative of f defined by

$$f(x, y, z) = e^{xyz}$$

at the point $(1, -1, 2)$ in the direction of the vector $\vec{u} = (1, 2, -2)$.

$D_{\vec{u^*}} f(1, -1, 2) =$

$\nabla f(x,y,z) = (yze^{xyz},\ xze^{xyz},\ xye^{xyz})$ $\quad\nabla f(1,-1,2) = (-2e^{-2}, 2e^{-2}, -e^{-2})$

$\vec{u^*} = \frac{(1,2,-2)}{\|(1,2,-2)\|} = \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$

$D_{\vec{u^*}} = (-2e^{-2}, 2e^{-2}, -e^{-2})\left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right) = \frac{4}{3}e^{-2}$

7.2 Gradient Vector in 2-D.

"In which direction $\hat{u}$ does $D_{\hat{u}}f(a,b)$ assume its largest value?"

angle between $\vec{u}$ & $\vec{v}$: $\|\vec{u}\|\|\vec{v}\|\cos\theta$.

- Greatest rate of change theorem (GRC)

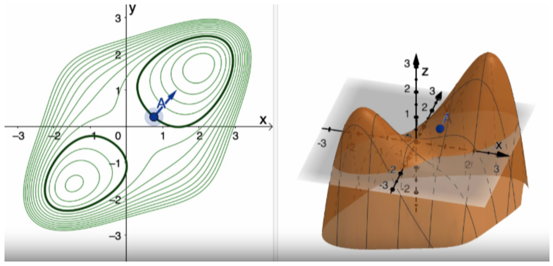
THEOREM 1 If $f(x,y)$ is differentiable at (a,b) and $\nabla f(a,b) \neq (0,0)$, then the largest value of $D_{\hat{u}}f(a,b)$ is $\|\nabla f(a,b)\|$, and occurs when $\hat{u}$ is in the direction of $\nabla f(a,b)$.

proof: $\because f$ is diff' at (a,b) $\quad \|\hat{u}\|=1$

$\therefore D_{\hat{u}}f(a,b) = \nabla f(a,b)\cdot\hat{u}$

$= \|\nabla f(a,b)\|\,\|\hat{u}\|\cos\theta$

$= \|\nabla f(a,b)\|\underbrace{\cos\theta}_{-1\sim 1}$.



Example 1

Find the largest rate of change of $f(x,y)=\sqrt{x^2+2y^2}$ at the point $(1,2)$ and the direction in which it occurs.

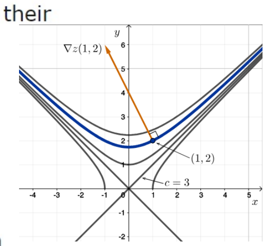
→ 相当于求 $\|\nabla f(1,2)\|$

$\nabla f(x,y) = \left(\dfrac{x}{\sqrt{x^2+2y^2}}, \dfrac{2y}{\sqrt{x^2+2y}}\right) \qquad \|\nabla f(1,2)\| = \|(\frac{1}{3},\frac{4}{3})\| = \frac{\sqrt{17}}{3}$

direction : $\hat{u} = \nabla f(1,2) = (\frac{1}{3},\frac{4}{3})$

Example 2

Let $z=f(x,y)=3-x^2+y^2$ represent the height above sea level. A hiker is at position $(1,2,6)$. In what direction should they move in order to follow a path of steepest ascent? What would be the slope of their path (i.e., rate of change of height with respect to horizontal distance)?



$\nabla f(x,y) = (-2x, 2y) \qquad \|\nabla f(1,2)\| = \sqrt{(-2)^2+4^2} = 2\sqrt{5}$

Example 3

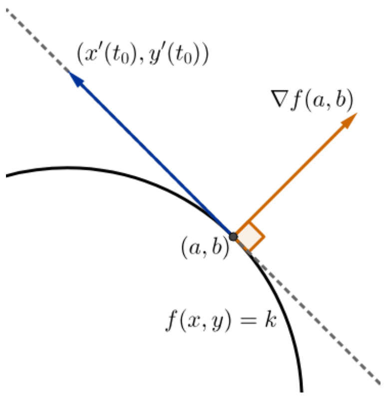
Let $f(x,y,z)=z^3e^{x^2+y^2-2x}$. Determine the greatest rate of change of f at $(1,1,1)$ and the direction in which it occurs.

$\nabla f(x,y,z) = \left(z^3(2x-2)e^{x^2+y^2-2x},\ z^3\cdot 2y\,e^{x^2+y^2-2x},\ 3z^2e^{x^2+y^2-2x}\right)$

$\underbrace{\nabla f(1,1,1) = (0,2,3)}_{\text{direction}} \qquad \underbrace{\|\nabla f(1,1,1)\| = \sqrt{13}}_{\text{greatest rate of change}}$

– Orthogonality theorem

THEOREM 2 If $f(x, y) \in C^1$ in a neighborhood of (a, b) and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = k$ through (a, b).



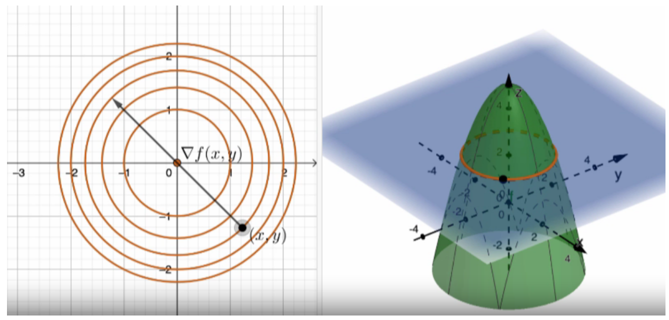
proof: $\because$ $f(x,y)$ 相当于沿着等高线走.

$\therefore \frac{df}{dt} = 0$

$$0 = \frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

$$= (f_x(x(t), y(t)),\ f_y(x(t), y(t))) \cdot (x'(t), y'(t))$$

$$= \nabla f(x, y) \cdot \vec{u} = 0$$



Your Turn

Prove that the level curves of the functions f and g defined by

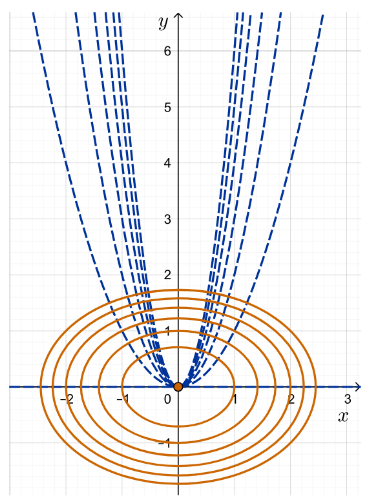
$$f(x, y) = \frac{y}{x^2}, \quad x \neq 0, \qquad g(x, y) = x^2 + 2y^2$$

intersect orthogonally. Sketch both sets of level curves to see their intersections.

↳ 相当于证 $\nabla f \cdot \nabla g = 0$

$\nabla f = \left(-\frac{2y}{x^3}, \frac{1}{x^2}\right)$ $\qquad \nabla g = (2x, 4y)$

$\nabla f \cdot \nabla g = \left(-\frac{2y}{x^3}, \frac{1}{x^2}\right) \cdot (2x, 4y) = 0$



7.3 The Gradient Vector in 3-D

- Orthogonality theorem in 3-D

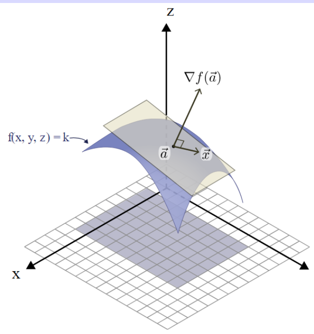
THEOREM 1 If $f(x, y, z) \in C^1$ in a neighborhood of (a, b, c) and $\nabla f(a, b, c) \neq (0, 0, 0)$, then $\nabla f(a, b, c)$ is orthogonal to the level surface $f(x, y, z) = k$ through (a, b, c).

$f(x, y, z) = k$

$\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$

$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$

↑ equation of tangent plane



Example 2

Find the equation of the tangent plane to the surface $z^3 e^{x^2+y^2-2x} = 1$ at the point $(1, 1, 1)$.

tangent plane is $\nabla f(1,1,1) \cdot (x-1, y-1, z-1) = 0$

$\nabla f(1,1,1) = (0, 2, 3)$

$(0, 2, 3)(x-1, y-1, z-1) = 0$

$2(y-1) + 3(z-1) = 0$

8.1 Taylor Polynomial of Degree 2

- Taylor polynomial of Single variable (1D)

$$P_{2,a}(x) = \underbrace{f(a) + f'(a)(x-a)}_{L_a(x)} + \frac{1}{2}\underbrace{f''(a)}_{P''_{2,a}(a) = f''(a)}(x-a)^2$$

- Taylor polynomial of 2 variable (2D)

DEFINITION

2nd degree Taylor polynomial

The **second degree Taylor polynomial** $P_{2,(a,b)}$ of $f(x, y)$ at (a, b) is given by

$$\underbrace{P_{2,(a,b)}}_{\text{近似值}}\underbrace{(x, y)}_{\text{真实值}} = \underbrace{f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)}_{L_{(a,b)}(x,y)}$$
$$+ \frac{1}{2}\left[f_{xx}(a, b)(x-a)^2 + 2f_{xy}(a, b)(x-a)(y-b) + f_{yy}(a, b)(y-b)^2\right]$$

General form of taylor polynomial:

$$P_{2,(a,b)}(x, y) = L_{(a,b)}(x,y) + A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2$$

$$2A = \frac{\partial^2 P_{2,(a,b)}}{\partial x^2} \qquad B = \frac{\partial^2 P_{2,(a,b)}}{\partial x \partial y} \qquad 2C = \frac{\partial^2 P_{2,(a,b)}}{\partial y^2}$$

DEFINITION

Hessian Matrix

The **Hessian matrix** of $f(x, y)$, denoted by $Hf(x, y)$, is defined as

$$Hf(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}$$

Example 1

Use the Taylor polynomial of degree 2 to approximate $\sqrt{(0.95)^3 + (1.98)^3}$.

$\rightarrow$ Let $f(x,y) = \sqrt{x^3+y^3}$ $\quad (a,b) = (1,2)$

$\rightarrow$ $\nabla f(x,y) = \left(\frac{1}{2}(x^3+y^3)^{-\frac{1}{2}} \cdot 3x^2, \ \frac{1}{2}(x^3+y^3)^{-\frac{1}{2}} \cdot 3y^2\right) = \left(\frac{1}{2}, 2\right)$

$f_{xx} = \frac{11}{12} \qquad f_{yy} = \frac{2}{3} \qquad f_{xy} = f_{yx} = -\frac{1}{3}$

$$Hf(1,2) = \begin{bmatrix} \frac{11}{12} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$\rightarrow$ $P_{2,(1,2)}(x,y) = 3 + \frac{1}{2}(x-1) + 2(y-2) + \frac{1}{2}\left[\frac{11}{12}(x-1)^2 - \frac{2}{3}(x-1)(y-2) + \frac{2}{3}(y-2)^2\right]$

$\sqrt{0.95^3 + 1.98^3} = P_{2,(1,2)}(0.95, 1.98) = 3 - 0.065 + \frac{1}{2} \cdot \frac{0.0227}{12} = 2.935946$

8.2 Taylor with 2nd degree remainder

- 1-D Taylor theorem

THEOREM 1 If $f''(x)$ exists on $[a, x]$, then there exists a number c between a and x such that

$$f(x) = f(a) + f'(a)(x - a) + R_{1,a}(x) \tag{8.2}$$

where

$$R_{1,a}(x) = \frac{1}{2} f''(c)(x - a)^2 \tag{8.3}$$

$L_a(x) = f(a) + f'(a)(x-a)$

Let $\underline{B \geq |f''(x)|}$ $\quad \forall x \in [a-\delta, a+\delta]$

$|f(x) - L_a(x)| = |R_{1,a}(x)|$

$= \left|\frac{1}{2} f''(c)(x-a)^2\right|$

$= \frac{1}{2} |f''(c)| (x-a)^2 \quad \leq \frac{1}{2} \underline{B} (x-a)^2$

- 2-D Taylor theorem

THEOREM 2 **(Taylor's Theorem)**

If $f(x, y) \in C^2$ in some neighborhood $N(a, b)$ of (a, b), then for all $(x, y) \in N(a, b)$ there exists a point (c, d) on the line segment joining (a, b) and (x, y) such that

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + R_{1,(a,b)}(x, y)$$

where

$$R_{1,(a,b)}(x, y) = \frac{1}{2}\left[f_{xx}(c, d)(x - a)^2 + 2 f_{xy}(c, d)(x - a)(y - b) + f_{yy}(c, d)(y - b)^2\right]$$

proof: $L(t) = (a + t(x-a),\ b + t(y-b))$

$f(x,y) = f(a,b) + f_x(a,b)h + f_y(a,b)k + R_{1,(a,b)}(x,y)$

$R_{1,(a,b)}(x,y) = \frac{1}{2}[f_{xx}(c,d)h^2 + 2 f_{xy}(c,d)hk + f_{yy}(c,d)k^2]$

Let $g(t) = f(L(t))$

$g(1) = g(0) + g'(0) + \frac{1}{2} g''(\tilde{c}) \quad (0 < \tilde{c} < 1) \qquad$ c always exist)

$\frac{1}{2} g''(\tilde{c}) = R_{1,(a,b)}(x,y)$

☆ taylor theo for $f(x,y)$ 只告诉我们存在 (c,d)
没有告诉我们如何找到

Example 1

Let $f(x,y)=\sqrt{1+x+2y}$. Find the linear approximation near $(0,0)$ and show that if $x\geq 0$ and $y\geq 0$, we have

$$|R_{1,(0,0)}(x,y)|\leq \frac{3}{4}(x^2+y^2)$$

→ Find linearization $\quad L_{(0,0)}(x,y)=f(0,0)+f_x(0,0)\,x+f_y(0,0)\,y.$

$f(0,0)=1 \qquad f_x(0,0)=\frac{1}{2} \qquad f_y(0,0)=1.$

$L_{(0,0)}(x,y)=1+\frac{1}{2}x+y.$

→ Find Hessian Matrix $\quad Hf(x,y)$

$$Hf(x,y)=\begin{bmatrix} \frac{-1}{4(1+x+2y)^{\frac{3}{2}}} & \frac{-1}{2(1+x+2y)^{\frac{3}{2}}} \\ \frac{-1}{2(1+x+2y)^{\frac{3}{2}}} & \frac{-1}{(1+x+2y)^{\frac{3}{2}}} \end{bmatrix}$$

→ Find error function

$$|R_{1,(0,0)}(x,y)|=\left|\frac{1}{2}\left[f_{xx}(c,d)x^2+2f_{xy}(c,d)xy+f_{yy}(c,d)y^2\right]\right|$$

(c,d) 未知，只能找 upper bound.

$$|f_{xx}(c,d)|=\left|-\frac{1}{4(1+c+2d)^{\frac{3}{2}}}\right|\leq\frac{1}{4}$$

$$|f_{xy}(c,d)|=\left|-\frac{1}{2(1+c+2d)^{\frac{3}{2}}}\right|\leq\frac{1}{2} \qquad \leftarrow \because x\geq 0 \;\; y\geq 0 \;\; \therefore c\geq 0 \;\; d\geq 0$$

$$|f_{yy}(c,d)|=\left|-\frac{1}{(1+c+2d)^{\frac{3}{2}}}\right|\leq 1$$

$$|R_{1,(0,0)}(x,y)|=\frac{1}{2}\Big[\underbrace{|f_{xx}(c,d)|}_{\leq\frac{1}{4}}x^2+2\underbrace{|f_{xy}(c,d)|}_{\leq\frac{1}{2}}|x||y|+\underbrace{|f_{yy}(c,d)|}_{\leq 1}y^2\Big]$$

$$\leq\frac{1}{2}\left[\frac{1}{4}x^2+2\cdot\frac{1}{2}|x||y|+y^2\right]$$

$$\leq\frac{1}{2}\left[\frac{1}{4}x^2+\frac{1}{2}(x^2+y^2)+y^2\right] \qquad (\text{since } 2|x||y|\leq x^2+y^2)$$

$$=\frac{3}{8}x^2+\frac{3}{4}y^2$$

COROLLARY 3 If $f(x,y)\in C^2$ in some closed neighborhood $N(a,b)$ of (a,b), then there exists a positive constant M such that

$$|R_{1,(a,b)}(x,y)|\leq M\|(x,y)-(a,b)\|^2, \quad \text{for all } (x,y)\in N(a,b)$$

8.3 Taylor 推广

In order to define the k-th degree Taylor polynomial for functions of two variables, we will need to introduce some notation.

If $f \in C^k$ is a function of n variables, we can write a k-th order partial derivative of $f(x_1, \ldots, x_n)$ as

$$\partial^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$$

where α is a **multi-index**; that is, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{N}$. The sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$ is called the **order** of α and is sometimes denoted by $|\alpha|$. We also define $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$.

Given a multi-index of order k, $\partial^\alpha f$ is a partial derivative of order k of f.

Your Turn 1

Let $f(x, y, z) = 2xz^3 e^{2y}$ and let $\alpha = (1, 4, 2)$. Determine $\partial^\alpha f$.

$$\partial^\alpha f = \left(\frac{\partial}{\partial x}\right)^1 \left(\frac{\partial}{\partial y}\right)^4 \left(\frac{\partial}{\partial z}\right)^2 f$$

$$\frac{\partial^2 f}{\partial z^2} = 12xz e^{2y}$$

$$\frac{\partial^4}{\partial y^4}\left(\frac{\partial^2 f}{\partial z^2}\right) = 2^4 \cdot 12xze^{2y} = 192xze^{2y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^4}{\partial y^4}\left(\frac{\partial^2 f}{\partial z^2}\right)\right) = 192ze^{2y}$$

In addition to using the multi-index notation for k-th order partial derivatives, we can also use it as follows.

Let $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{a} = (a_1, a_2, \ldots, a_n)$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then

$$(\vec{x} - \vec{a})^\alpha = (x_1 - a_1)^{\alpha_1} (x_2 - a_2)^{\alpha_2} \cdots (x_n - a_n)^{\alpha_n}$$

Your Turn 2

Let $\vec{x} = (x, y, z, w)$, $\vec{a} = (-5, 3, 7, -2)$, and $\alpha = (3, 2, 4, 4)$.

Determine $(\vec{x} - \vec{a})^\alpha$.

$$(\vec{x} - \vec{a})^\alpha = (x+5)^3 (y-3)^2 (z-7)^4 (w+2)^4$$

- k-th degree Taylor polynomial

The **k-th degree Taylor polynomial** of a function $f(x,y)$ is

$$P_{k,(a,b)}(x,y) = \sum_{|\alpha|\le k} \partial^\alpha f(a,b) \frac{[(x,y)-(a,b)]^\alpha}{\alpha!}$$

THEOREM 1

Taylor's Theorem of order k
If $f(x,y) \in C^{k+1}$ at each point on the line segment joining (a,b) and (x,y), then there exists a point (c,d) on the line segment between (a,b) and (x,y) such that

$$f(x,y) = P_{k,(a,b)}(x,y) + R_{k,(a,b)}(x,y)$$

where

$$R_{k,(a,b)}(x,y) = \frac{1}{(k+1)!}\left[(x-a)D_1 + (y-b)D_2\right]^{k+1} f(c,d)$$

Example 3

Write out $P_{2,(a,b)}(x,y)$ using the subscript notation for partial derivatives.

$$P_{2,(a,b)}(x,y) = \sum_{|\alpha|\le 2} \partial^\alpha f(a,b) \frac{[(x,y)-(a,b)]^\alpha}{\alpha!}$$

$$P_{2,(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2} f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2} f_{yy}(a,b)(y-b)^2$$

COROLLARY 2

If $f(x,y) \in C^{k+1}$ in some closed neighborhood $N(a,b)$ of (a,b), then there exists a constant $M > 0$ such that

$$|f(x,y) - P_{k,(a,b)}(x,y)| \le M\,\|(x,y)-(a,b)\|^{k+1}$$

for all $(x,y) \in N(a,b)$.

COROLLARY 3

If $f(x,y) \in C^{k+1}$ in some neighborhood of (a,b), then

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - P_{k,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|^k} = 0$$

9.1 Local Extrema and Critical Points

- Critical Points in 1 variable

def. critical point. $f'(c)=0$ or $f'(c)$ undefined

└→ 用于找 local extrema

If $f(x)$ has a critical point $x=c$, then $f(x)$ has a local extremum at $x=c$.

- (●) True
- ( ) False

counter-example $f(x)=x^3$. critical point: $x=0$ → not local max/min

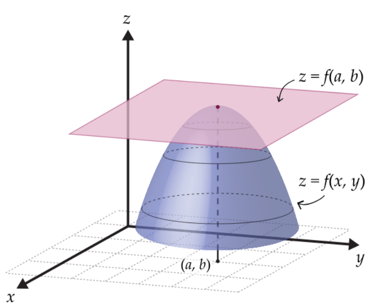
- Critical Points in 2 variables

DEFINITION

Local Maximum and Minimum

A point (a,b) is a **local maximum point of** f if $f(x,y) \le f(a,b)$ for all (x,y) in some neighborhood of (a,b).

A point (a,b) is a **local minimum point of** f if $f(x,y) \ge f(a,b)$ for all (x,y) in some neighborhood of (a,b).



假设 (a,b) is local max/min for $f(x,b)$ & $f(a,y)$ in 横截面

(a,b) is a critical point of cross-sections $f(x,b)$ & $f(a,y)$

假设 f has partial derivatives. $\frac{\partial f}{\partial x}(a,b)=0$ $\frac{\partial f}{\partial y}(a,b)=0$

∴ equation of tangent plane at (a,b) is $z=f(a,b)$

THEOREM 1

If (a,b) is a local maximum or minimum point of f, then

$$f_x(a,b) = 0 = f_y(a,b)$$

相当于平面平行于 $(x,y)=(0,0)$

or at least one of f_x or f_y does not exist at (a,b).

DEFINITION

Critical Point

A point (a,b) in the domain of $f(x,y)$ is called a **critical point** of f if $\frac{\partial f}{\partial x}(a,b)=0$ or $\frac{\partial f}{\partial x}(a,b)$ does not exist, and $\frac{\partial f}{\partial y}(a,b)=0$ or $\frac{\partial f}{\partial y}(a,b)$ does not exist.

Example 1

Find the critical points of $f(x,y) = x^2 + y^2$ and determine whether they are local maxima or local minima.

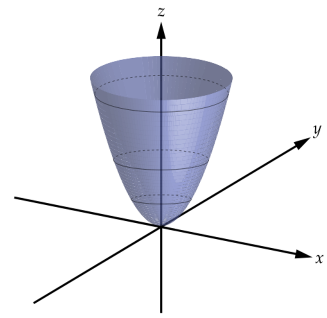
$f_x(x,y) = 2x \qquad f_y(x,y) = 2y.$

$\because f_x(x,y) = 0$ and $f_y(x,y) = 0$ when $x = 0$, $y = 0$

$\therefore (0,0)$ is the only critical point of f.

$f(x,y) = x^2 + y^2 \geq 0 = f(0,0) \quad \forall (x,y) \neq (0,0)$

$\therefore (0,0)$ is local min.



Example 2

Find the critical points of $g(x,y) = -x^2 - y^2$ and determine whether they are local maxima or local minima.

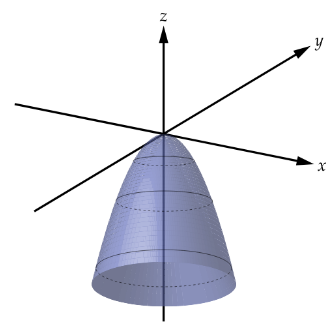
$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$

$\because f_x(x,y) = 0$ and $f_y(x,y) = 0$ when $x = 0$, $y = 0$

$\therefore (0,0)$ is the only critical point of f.

$f(x,y) = x^2 + y^2 \leq 0 = f(0,0) \quad \forall (x,y) \neq (0,0)$

$\therefore (0,0)$ is local max.



Example 3

Find the critical points of $h(x,y) = x^2 - y^2$ and determine whether they are local maxima or local minima.

$f_x(x,y) = 2x \qquad f_y(x,y) = -2y$

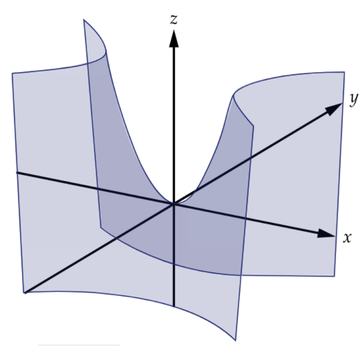
$\because f_x(x,y) = 0$ and $f_y(x,y) = 0$ when $x = 0$, $y = 0$

$\therefore (0,0)$ is the only critical point of f.

This time, we have $h(x,0) > h(0,0) \quad \forall x.$

$h(0,y) < h(0,0) \quad \forall y.$

$\therefore (0,0)$ is neither a local max or local min.

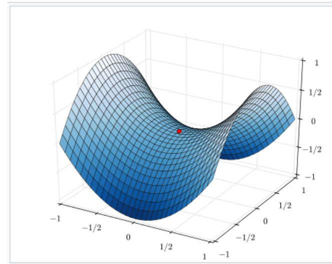


DEFINITION
Saddle Point

A critical point (a, b) of $f(x, y)$ is called a **saddle point** of f if in every neighborhood of (a, b) there exist points (x_1, y_1) and (x_2, y_2) such that 鞍点

$$f(x_1, y_1) > f(a, b) \text{ and } f(x_2, y_2) < f(a, b)$$

沿一个轴向(峰值之间)具有 相对 min /max 临界点 in 交叉轴



A saddle point (in red) on the graph of $z = x^2 - y^2$ (hyperbolic paraboloid)

Example 4

Find all critical points of $f(x, y) = x^2y + 3xy^2 + xy$.

$\frac{\partial f}{\partial x} = 2xy + 3y^2 + y$ $\quad$ $\frac{\partial f}{\partial y} = x^2 + 6xy + x$

$2xy + 3y^2 + y^2 = 0$ $\quad$ $y(2x + 3y + 1) = 0$ (*) $\quad$ $y = 0$ $\quad$ $2x + 3y + 1 = 0$

$x^2 + 6xy + x = 0$ $\quad$ $x(x + 6y + 1) = 0$ (**) $\quad$ $x = 0$ $\quad$ $x + 6y + 1 = 0$

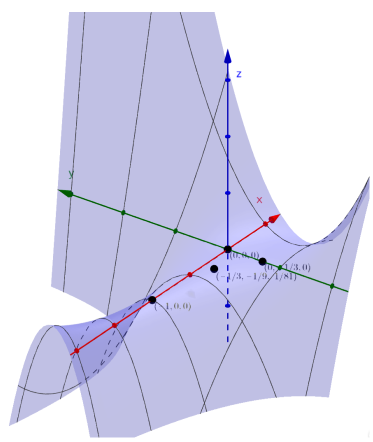
case 1 : $y = 0$ 代入 (**)↓

$x(x + 6y + 1) = 0$ $\quad$ $x(x + 6 \cdot 0 + 1) = 0$

$x(x + 1) = 0$

$x = 0$ $\quad$ $x = -1$

critical point : $(0, 0)$ $\quad$ $(-1, 0)$



case 2 : $2x + 3y + 1 = 0$

↳ $y = \frac{-2x - 1}{3}$ 代入 (**)

$x(x + 6y + 1) = 0$ $\Rightarrow$ $x\left(x + 6 \cdot \left(\frac{-2x-1}{3}\right) + 1\right) = 0$

$x(-3x - 1) = 0$

$x = 0$ $\quad$ $x = -\frac{1}{3}$

critical point : $(0, -\frac{1}{3})$ $\quad$ $(-\frac{1}{3}, -\frac{1}{9})$

9.2 Second Derivative Test

- f'' for one variable

second degree Taylor Polynomial : $(x \to a)$ $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$

If $x=a$ is critical point of f, then $f'(a)=0$

$$f(x) = f(a) + \frac{1}{2} f''(a)(x-a)^2$$

$$f(x) - f(a) = \frac{1}{2} f''(a)(x-a)^2$$

$$\begin{cases} f''(a) > 0 & f(x)-f(a) > 0 & x=a \text{ 是 local min} \\ f''(a) = 0 & \text{无法得出结论} & \\ f''(a) < 0 & f(x)-f(a) < 0 & x=a \text{ 是 local max} \end{cases}$$

- Quadratic form

DEFINITION
Quadratic Form
on $\mathbb{R}^2$

A function Q of the form

$$Q(u, v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$$

where a_{11}, a_{12} and a_{22} are constants, is called a **quadratic form** on $\mathbb{R}^2$.

$$Q(u,v) = [u \quad v] \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}}_{\text{symmetric}} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{cases} \forall (u,v) \neq (0,0) \quad Q(u,v) > 0 & \text{positive definite} & \text{local max} & \det > 0 & a_{11} > 0 \\ \forall (u,v) \neq (0,0) \quad Q(u,v) < 0 & \text{negative definite} & \text{local min} & \det > 0 & a_{11} < 0 \\ \exists (u,v) \quad Q(u,v) < 0 \quad \exists (w,z) \quad Q(w,z) > 0 & \text{indefinite} & \text{saddle} & \det < 0 & \\ \text{其它} \quad Q \text{ is semifinite} \quad \det = 0 & \begin{cases} \forall (u,v) \quad Q(u,v) \geq 0 & \text{positive semifinite} \\ \forall (u,v) \quad Q(u,v) \leq 0 & \text{negative semifinite} \end{cases} & & & \end{cases}$$

Q Classify the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$.

associated quadratic form : $Q(u,v) = u^2 + 6uv + 2v^2$
$= (u+3v)^2 - 7v^2$ 通过中间的数去 complete the square

$\because Q(u,0) = u^2 > 0 \ (u \neq 0) \quad Q(-3v, v) = -7v^2 < 0 \quad v \neq 0$

$\therefore$ indefinite

THEOREM 1 **Second Partial Derivatives Test**

Suppose that $f(x, y) \in C^2$ in some neighborhood of (a, b) and that

$$f_x(a, b) = 0 = f_y(a, b)$$

(1) If $Hf(a, b)$ is positive definite, then (a, b) is a local minimum point of f.

(2) If $Hf(a, b)$ is negative definite, then (a, b) is a local maximum point of f.

(3) If $Hf(a, b)$ is indefinite, then (a, b) is a saddle point of f.

$(x,y) \to (a,b)$

$$f(x,y) \approx P_{2,(a,b)}(x,y)$$
$$\approx f(a,b) + f_x \cdot (x-a) + f_y \cdot (y-b) + \frac{1}{2}\left[f_{xx}(x-a)^2 + 2f_{xy} \cdot (x-a)(x-b) + f_{yy} \cdot (y-b)^2\right]$$

(a,b) : critical point 二阶导仅适用于 local max/min.

当 critical point 为 local max/min 时 $f_x(a,b) = 0 = f_y(a,b)$

$\therefore$ 上式可化简为:

$$f(x,y) - f(a,b) \approx \frac{1}{2}\left[f_{xx}(x-a)^2 + 2f_{xy} \cdot (x-a)(x-b) + f_{yy} \cdot (y-b)^2\right]$$

$\hookrightarrow f(x,y) - f(a,b) > 0 \Rightarrow Hf(a,b)$ is positive definite

- determinant & quadratic forms

THEOREM 2 If $Q(u, v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$ and $D = a_{11}a_{22} - a_{12}^2$, then

(1) Q is positive definite if and only if $D > 0$ and $a_{11} > 0$

(2) Q is negative definite if and only if $D > 0$ and $a_{11} < 0$

(3) Q is indefinite if and only if $D < 0$

(4) Q is semidefinite if and only if $D = 0$

Example 3

Classify the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$.

$\det(A) = 2 - 9 = -7 < 0$ indefinite

Example 4

Find and classify all critical points of the function $f(x, y) = x^3 - 4x^2 + 4x - 4xy^2$.

$\to f_x(x,y) = 3x^2 - 8x + 4 - 4y^2 = 0$ ①

$f_y(x,y) = -8xy = 0$ ②

$x = 0$ 代入① $0 = 4 - 4y^2$ $y = \pm 1$

$y = 0$ 代入① $0 = 3x^2 - 8x + 4 = (3x-2)(x-2)$ $x = 2$ $x = \frac{2}{3}$

critical point : $(0,1)$ $(0,-1)$ $(2,0)$ $(\frac{2}{3},0)$

$\to f_{xx}(x,y) = 6x - 8$ $f_{xy}(x,y) = -8y$ $f_{yy}(x,y) = -8x$

$Hf(0,1) = \begin{bmatrix} -8 & -8 \\ -8 & 0 \end{bmatrix}$ det $= -64 < 0$ saddle

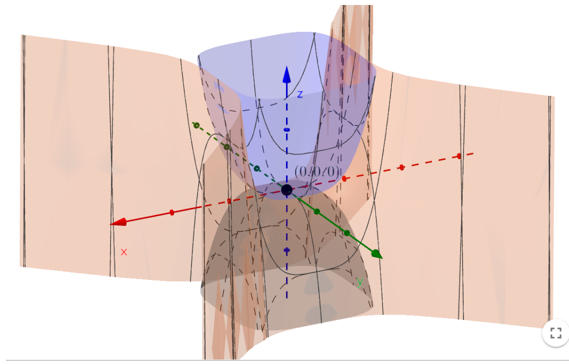
$Hf(0,-1) = \begin{bmatrix} -8 & 8 \\ 8 & 0 \end{bmatrix}$ det $= -64 < 0$ saddle

$Hf(2,0) = \begin{bmatrix} 4 & 0 \\ 0 & -16 \end{bmatrix}$ det $= -64 < 0$ saddle

$Hf(\frac{2}{3},0) = \begin{bmatrix} -4 & 0 \\ 0 & -\frac{16}{3} \end{bmatrix}$ det $= \frac{64}{3} > 0$ local min

– Degenerate Critical Points

$f(x,y) = x^4 + y^4$ $\quad$ $g(x,y) = x^4 - y^4$ $\quad$ $h(x,y) = -x^4 - y^4$



consider point $(0,0)$

$f(x,y) - f(0,0) \geq 0 \rightarrow$ local min

$g(x,0) - g(0,0) \geq 0 \quad \forall x$

$g(0,y) - g(0,0) \leq 0 \quad \forall y$

$\rightarrow$ saddle

$h(x,y) - h(0,0) \leq 0 \rightarrow$ local max

def. degenerate

$Hf(a,b)$ is semidefinite

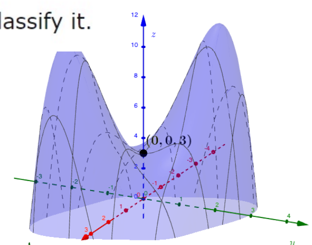
second partial derivative test 无法得出结论

$Hf(a,b)$ semidefinite. $\Rightarrow (a,b)$ degenerate critical point.

$f(x,y) - f(a,b)$
- $\rightarrow$ always $+$ $\quad$ local min
- $\rightarrow$ always $-$ $\quad$ local max
- $\rightarrow \exists + \exists -$ $\quad$ saddle

Example 5

Show that $(0,0)$ is a degenerate critical point of $f(x,y) = 2(x-y)^2 - x^4 - y^4 + 3$ and classify it.



$\rightarrow \nabla f(0,0) = (0,0)$ $\quad$ $Hf(0,0) = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$

$Q(u,v) = 4u^2 - 8uv + 4v^2 = 4(u-v)^2 \geq 0$

$\forall u \quad Q(u,u) = 0. \quad \Rightarrow Hf(0,0)$ semi. $\quad (0,0)$ degenerate critical point

$\rightarrow f(x,y) - f(0,0) = 2(x-y)^2 - x^4 - y^4$

$f(x,x) - f(0,0) = -2x^4 < 0$ $\quad$ $\because f(x,y) - f(0,0) \; \exists + \exists -$

$f(x,0) - f(0,0) = x^2(2 - x^2) > 0$ $\quad$ $\therefore (0,0)$ saddle point

Let's summarize what we learned about critical points and the second derivative test for functions of two variables.

To find and classify the critical points of a function of two variables $f(x, y)$:

1. Find the critical points of f. These are the points (a, b) such that

$$\frac{\partial f}{\partial x}(a,b) = 0 \quad \text{or} \quad \frac{\partial f}{\partial x}(a,b) \quad \text{DNE} \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b) = 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(a,b) \quad \text{DNE}$$

2. Classify the critical points of $f(x, y)$ by calculating $Hf(a, b)$:
   - If $Hf(a, b)$ is negative definite, then (a, b) is a local maximum
   - If $Hf(a, b)$ is indefinite, then (a, b) is a saddle point
   - If $Hf(a, b)$ is positive definite, then (a, b) is a local minimum
   - If $Hf(a, b)$ is semidefinite, then (a, b) is a degenerate critical point. Check the sign of $f(x, y) - f(a, b)$ in a small neighbourhood of (a, b):
     - If the sign of $f(x, y) - f(a, b)$ is always positive, then (a, b) is a local minimum
     - If the sign of $f(x, y) - f(a, b)$ is always negative, then (a, b) is a local maximum
     - If $f(x, y) - f(a, b)$ assumes both positive and negative values, then (a, b) is a saddle point

原理: $f \in C^2$ (a,b) is critical pt.

$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1(a,b)}(x,y)$

By Taylor's theorem. $R_{1(a,b)}(x,y) = \frac{1}{2}[f_{xx}(c,d)(x-a) + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2]$

$R_{1(a,b)}(x,y) > 0 \Rightarrow f(a,b) < f(x,y) \Rightarrow (a,b)$ is local min

$R_{1(a,b)}(x,y) < 0 \Rightarrow f(a,b) > f(x,y) \Rightarrow (a,b)$ is local max

$R_{1(a,b)}(x,y) >\&< 0 \Rightarrow f(a,b) >\&< f(x,y) \Rightarrow (a,b)$ is saddle.

$R_{1(a,b)}(x,y) = \frac{1}{2}[\underbrace{f_{xx}(c,d)}_{A}\underbrace{(x-a)}_{u^2} + 2\underbrace{f_{xy}(c,d)}_{B}\underbrace{(x-a)}_{u}\underbrace{(y-b)}_{v} + \underbrace{f_{yy}(c,d)}_{C}\underbrace{(y-b)^2}_{v^2}]$

$= \frac{1}{2}\begin{bmatrix} u & v \end{bmatrix}\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

9.3 Convex functions 凸形函数

- Convex function of one variable

Definition: Convex and strictly convex functions of one variable

A twice differentiable function $f(x)$ is **convex** if $f''(x) \geq 0$ for all x and f is **strictly convex** if $f''(x) > 0$ for all x. Thus the term convex means "concave up".

Properties of convex functions of one variable:

THEOREM 3 If $f(x)$ is twice continuously differentiable and strictly convex, then

(1) $f(x) > L_a(x) = f(a) + f'(x)(x-a)$ for all $x \neq a$, for any $a \in \mathbb{R}$.

(2) For $a < b$, $f(x) < f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ for $x \in (a,b)$.

- Convex function of two variables

Definition: Convex and strictly convex functions of two variables

Let $f(x,y)$ have continuous second partial derivatives. We say that f is **convex** if $Hf(x,y)$ is positive semi-definite for all (x,y) and that f is **strictly convex** if $Hf(x,y)$ is positive definite for all (x,y).

Properties of convex functions of two variables:

THEOREM 4 If $f(x,y)$ has continuous second partial derivatives and is strictly convex, then

(1) $f(x,y) > L_{(a,b)}(x,y)$ for all $(x,y) \neq (a,b)$, and

(2) $f(a_1+t(b_1-a_1), a_2+t(b_2-a_2)) < f(a_1,a_2)+t[f(b_1,b_2)-f(a_1,a_2)]$ for $0 < t < 1$, $(a_1,a_2) \neq (b_1,b_2)$.

10.1 Extreme Value Theorem

- absolute maximum and minimum $f(x, y)$

DEFINITION

Absolute Maximum and Minimum

Given a function $f(x, y)$ and a set $S \subseteq \mathbb{R}^2$,

1. a point $(a, b) \in S$ is an **absolute maximum point** of f on S if

$$f(x, y) \leq f(a, b) \quad \text{for all } (x, y) \in S$$

   The value $f(a, b)$ is called the **absolute maximum value** of f on S.

2. a point $(a, b) \in S$ is an **absolute minimum point** of f on S if

$$f(x, y) \geq f(a, b) \quad \text{for all } (x, y) \in S$$

   The value $f(a, b)$ is called the **absolute minimum value** of f on S.

- EVT of $f(x)$

THEOREM 1 **(The Extreme Value Theorem)**

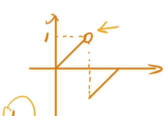
If $f(x)$ is continuous on a finite closed interval I, then there exists $c_1, c_2 \in I$ such that

$$f(c_1) \leq f(x) \leq f(c_2) \quad \text{for all } x \in I$$

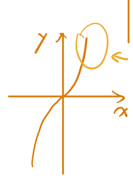
EXERCISE 1 Give a function $f(x)$ and an interval I such that

1. I is closed, but f does not have an absolute maximum on I.
2. I is finite and f is continuous on I, but f does not have an absolute maximum on I.
3. I is infinite and f is continuous on I, but f does not have an absolute minimum.

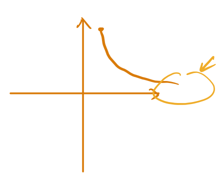
1. $I = [0, 2]$ $\quad f(x) = \begin{cases} x & 0 \leq x < 1 \\ x-2 & 1 \leq x \leq 2 \end{cases}$



2. $I = (0, \frac{\pi}{2})$ $\quad f(x) = \tan(x)$

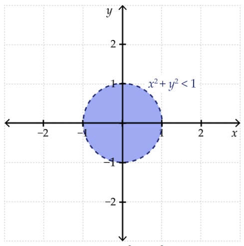


3. $I = [1, \infty)$ $\quad f(x) = \frac{1}{x}$

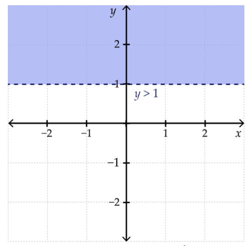


DEFINITION
Bounded Set

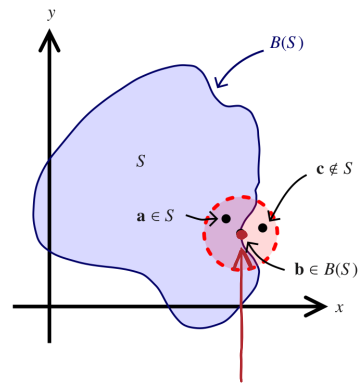
A set $S \subset \mathbb{R}^2$ is said to be **bounded** if and only if it is contained in some neighbourhood of the origin.



bounded



unbounded



DEFINITION
Boundary Point

Given a set $S \subset \mathbb{R}^2$, a point $(a, b) \in \mathbb{R}^2$ is said to be a **boundary point** of S if and only if every neighbourhood of (a, b) contains at least one point in S and one point not in S.

DEFINITION
Boundary of S

The set $B(S)$ of all boundary points of S is called the **boundary** of S.

DEFINITION
Closed Set

A set $S \subseteq \mathbb{R}^2$ is said to be **closed** if S contains all of its boundary points.

相当于 boundary point 的集合　　所有涉及到"<"的都 not closed

Your Turn 4

Determine which of the following sets in $\mathbb{R}^2$ is closed:

- $S = \{(x, y) \in \mathbb{R}^2 \mid 3x + 4y + 1 = 0\}$ — Closed / Not closed
- $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 9\}$ — Closed / Not closed
- $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$ — Closed / Not closed
- $S = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ — Closed / Not closed
- $S = \mathbb{R}^2$ — Closed / Not closed

The set $\mathbb{R}^2$ is closed since it doesn't have a boundary thus contains all of its boundary points

THEOREM 2 If $f(x,y)$ is continuous on a closed and bounded set $S \subset \mathbb{R}^2$, then there exists points $(a,b),(c,d) \in S$ such that

$$f(a,b) \le f(x,y) \le f(c,d) \qquad \text{for all } (x,y) \in S$$

☆ a closed and bounded set 永远有 absolute max & min

Your Turn

Determine in which cases the EVT for functions of two variables applies.

- $S = \mathbb{R}^2$ and $f(x,y) = x^2 + y^2$ — EVT applies / EVT does not apply ✓

∵ S is not bounded, ← $x/y \uparrow$ $f(x,y) \uparrow$

∴ EVT doesn't apply

Example 4

Let $S = \{(x,y) \in \mathbb{R}^2 \mid x > -1, y \in \mathbb{R}\}$ and let $f(x,y) = \begin{cases} 1 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

∵ $f(x,y)$ not continuous

∴ EVT doesn't apply

☆ 即使不满足 EVT，$f(x,y)$ 也可能有 absolute max/min

10.2 Algorithm for extreme values

- Closed interval method

$f(x)$ 在 $[a,b]$ 的极值 $\begin{cases} ① \text{端点} \\ ② f'(c)=0 \\ ③ f'(c) \text{ DNE} \end{cases}$

$f(a,b)$ 在 $B(S)$ 的极值 $\begin{cases} ① \text{ critical point in } S \\ ② \text{ boundary set on } S \end{cases}$

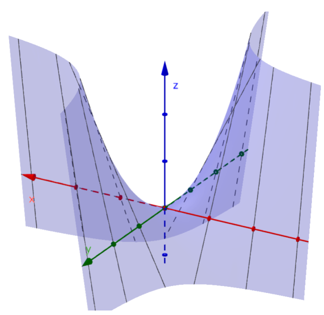
ALGORITHM

Let $S \subset \mathbb{R}^2$ be closed and bounded. To find the maximum and/or minimum value of a function $f(x, y)$ that is continuous on S,

(1) Find all critical points of f that are contained in S. Evaluate f at each such point.

(2) Find the maximum and minimum points of f on the boundary $B(S)$.

(3) The maximum value of f on S is the largest value of the function found in steps (1) and (2). The minimum value of f on S is the smallest value of the function found in steps (1) and (2).



Example 1

Find the maximum value of $f(x, y) = xy$ on the set

$$S = \{(x, y) \mid x^2 + y^2 \le 1\}$$

→ $\because \nabla f(x,y) = (y, x)$

$\therefore$ the only critical point of f is $(0,0) \in S$. $f(0,0)=0$

→ 找 max & min.

当边界为直线时，用x/y中一个求解另一个

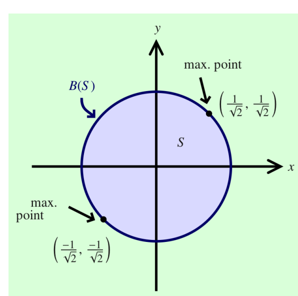
当边界为圆形时，使用参数方程.

① critical point

Let $x=\cos t$, $y=\sin t$ $(0 \le t \le 2\pi)$ $g(t) = f(\cos t, \sin t) = \cos t \sin t = \frac{1}{2}\sin 2t$

$g'(t) = \cos 2t$ $\therefore$ critical point: $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

$g(\frac{\pi}{4}) = \frac{1}{2}$ $g(\frac{3\pi}{4}) = -\frac{1}{2}$ $g(\frac{5\pi}{4}) = \frac{1}{2}$ $g(\frac{7\pi}{4}) = -\frac{1}{2}$



② boundary set $g(0)=0$ $g(2\pi)=0$

→ 比较 $g(x)$ 的值, 得出 max point

最大值为 $\frac{1}{2}$. 对应的点为 $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Your Turn

Find the maximum of $f(x,y)=x^2y-y$ on the set $S=\{(x,y)\mid 9x^2+4y^2\le 36\}$ if it exists.

→ f is continuous on S, and S is closed and bounded

∴ EVT can apply

→ 列出 critical points

$\nabla f(2xy,\ x^2-1)=0$

critical point $(1,0)$ $(-1,0)$

→ 写出 boundary set

$B(S)=\{(x,y)\mid 9x^2+4y^2=36\}$

→ Let $x=2\cos t$ $y=3\sin t$

$g(t)=f(2\cos t,\ 3\sin t)$

$=3\sin t\,(4\cos^2 t-1)$

$=3\sin t\,(4-4\sin^2 t-1)$

$=3\sin t\,(3-4\sin^2 t)$

$g'(t)=3\cos t(3-4\sin^2 t)-8\sin t\cos t\cdot 3\sin t$

$=9\cos t\,(1-4\sin^2 t)=0$

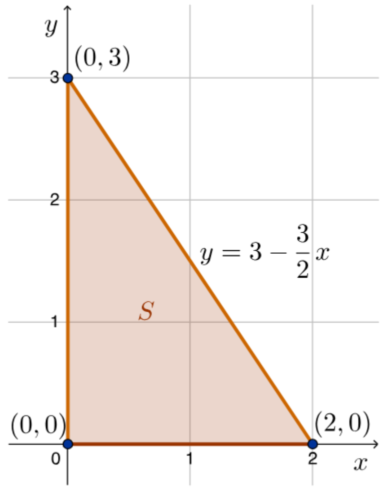
critical point $t=\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$

$-3 \quad 3 \quad 3 \quad 3 \quad -3 \quad -3$

$(0,-3)$ $(\sqrt{3},\frac{3}{2})$ $(-\sqrt{3},\frac{3}{2})$

Example 2

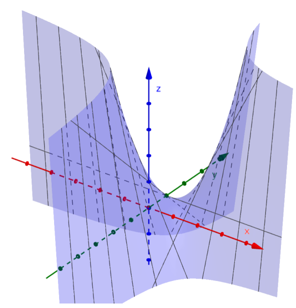
Find the maximum and minimum value of $f(x,y) = xy - 2x - y + 2$ on the triangular region S with vertices $(0,0)$, $(2,0)$ and $(0,3)$.



→ check critical point

∵ $f(x,y) = (y-2,\ x-1)$ ∴ only critical point is $(1,2)$.

∵ $(1,2) \notin S$ ∴ solution 不可能是 $(1,2)$



→ check boundary point

① $x=0$. $0 \le y \le 3$.

$g(y) = f(0,y) = -y+2$. by observation, local extrema $(0,0)$ $(0,3)$

② $y=0$. $0 \le x \le 2$

$g(x) = f(x,0) = -2x+2$. by observation, local extrema $(0,0)$ $(2,0)$

③ $y = 3 - \frac{3}{2}x$. $0 \le x \le 2$

$g(x) = f(x, 3-\frac{3}{2}x) = -\frac{3}{2}x^2 + \frac{5}{2}x - 1$. $g'(x) = -3x + \frac{5}{2} = 0$. $x = \frac{5}{6}$.

local extrema $(\frac{5}{6}, \frac{7}{4})$

→ evaluate

$f(0,0) = 2$ $f(0,3) = -1$ $f(2,0) = -2$ $f(\frac{5}{6}, \frac{7}{4}) = \frac{1}{24}$

max min

10.3 Optimization with constraints

- 实际问题. 找到约束 $g(x,y)=k$ 的函数 $f(x,y)$ 的最大/最小值.

Q. 三条产品线: x, y, z. 利润: $P(x,y,z) = ax+by+cz$

生产成本维持在 k 美元 $C(x,y,z)=k$.

希望利益 $P(x,y,z)$ 最大化.

不用参数化曲线得到最大/最小值.

- Lagrange Multipliers

找 max/min value of $f(x,y)$. constraint: $g(x,y)=k$. $g \in C^1$

$\equiv$ find max/min of $f(x,y)$ level curve: $g(x,y)=k$.

2 Variables

ALGORITHM (Lagrange Multiplier Algorithm)

Assume that $f(x, y)$ is a differentiable function and $g \in C^1$. To find the maximum value and minimum value of f subject to the constraint $g(x, y) = k$, evaluate $f(x, y)$ at all points (a, b) which satisfy one of the following conditions.

(1) $\nabla f(a,b) = \lambda \nabla g(a,b)$ and $g(a,b) = k$. λ: lagrange multiplier

(2) $\nabla g(a,b) = (0,0)$ and $g(a,b) = k$.

(3) (a,b) is an end point of the curve $g(x,y) = k$.

The maximum/minimum value of $f(x, y)$ is the greatest/least value of f obtained at the points found in (1)-(3).

解 $\nabla f_x(x,y) = \lambda \nabla g_x(x,y)$

$\nabla f_y(x,y) = \lambda \nabla g_y(x,y)$

$g(x,y) = k$

※ $\nabla g(a,b) \neq (0,0)$

* 若 $g(x,y)=k$ unbounded. 对 $\lim_{\|(x,y)\|\to\infty} f(x,y)$, (x,y) 满足 $g(x,y)=k$.

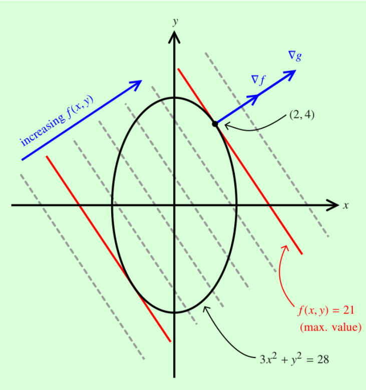
Example 1

Find the maximum value of $6x+4y-7$ on the ellipse $3x^2+y^2=28$.

→ check condition $\nabla f(x,y)=\lambda\nabla g(x,y)$ $\quad g(x,y)=3x^2+y^2=28$

$f_x=6 \quad f_y=4 \quad \nabla f(x,y)=(6,4)$

$g_x=6x \quad g_y=2y \quad \nabla g(x,y)=(6x,2y)$



解 $\boxed{\begin{array}{c} 6=6\lambda x \\ 4=2\lambda y \\ 3x^2+y^2=28 \end{array}}$

obtain points : $(2,4)$ $\quad(-2,-4)$

→ check condition $\nabla g(x,y)=(0,0)$ $\quad g(x,y)=28$.

$\because (0,0)=\nabla g(x,y)=(6x,2y) \Rightarrow x=y=0$. 不满足 constraints

$\therefore$ No points

→ check endpoints.

$\because$ constrains is closed curve

$\therefore$ No endpoints

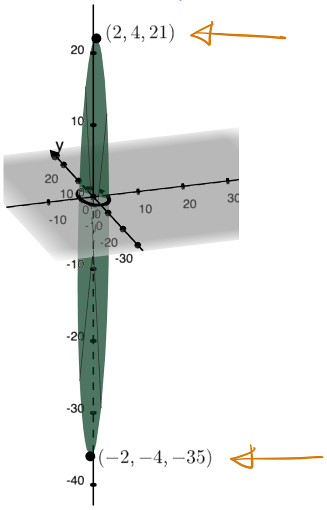
找点

比较

→ evaluate

$f(2,4)=21 \quad f(-2,-4)=-35$

$\therefore$ max of f on $3x^2+y^2=28$. is 21. occurs at $(2,4)$



Example 2

Find the maximum and minimum values of $f(x,y)=y$ on the curve defined by $y^2+x^4-x^3=0$.

$$\boxed{\begin{aligned} &\max \quad f(x,y)=y \\ &\text{s.t.} \quad g(x,y)=y^2+x^4-x^3=0 \end{aligned}}$$

→ check condition $\nabla f(x,y)=\lambda\nabla g(x,y)$ $\quad g(x,y)=y^2+x^4-x^3=0$

$f_x=0 \qquad f_y=1 \qquad \nabla f(x,y)=(0,1)$

$g_x=4x^3-3x^2 \qquad g_y=2y \qquad \nabla g(x,y)=(4x^3-3x^2,2y)$

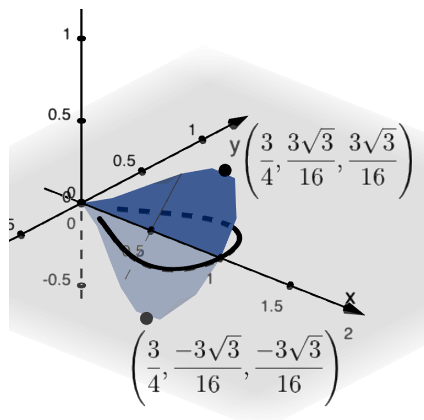
解 $\boxed{\begin{aligned} &0=\lambda(4x^3-3x^2) \\ &1=\lambda 2y \\ &y^2+x^4-x^3=0 \end{aligned}}$ $\rightarrow \lambda\neq 0$

$\lambda=\frac{1}{2y} \qquad \frac{1}{2y}\cdot x^2(4x-3)=0$

$x=0 \qquad y=0$

$x=\frac{3}{4} \qquad y=\pm\frac{3\sqrt{3}}{16}$

$\therefore$ points $\left(\frac{3}{4},\frac{3\sqrt{3}}{16}\right)$ $\left(\frac{3}{4},-\frac{3\sqrt{3}}{16}\right)$



→ check condition $\nabla g(x,y)=(0,0)$ $\quad g(x,y)=0$

$(0,0)=\nabla g(x,y)=(4x^3-3x^2,2y)$

→ check endpoints.

$\because$ constrains is closed curve (通过画图得出)

$\therefore$ No endpoints

→ evaluate

$f\left(\frac{3}{4},-\frac{3\sqrt{3}}{16}\right)=\underset{\text{min}}{\underline{-\frac{3\sqrt{3}}{16}}} \qquad f\left(\frac{3}{4},\frac{3\sqrt{3}}{16}\right)=\underset{\text{max}}{\underline{\frac{3\sqrt{3}}{16}}} \qquad f(0,0)=0$

max point $\left(\frac{3}{4},\frac{3\sqrt{3}}{16},\frac{3\sqrt{3}}{16}\right)$ $\qquad$ min point $\left(\frac{3}{4},-\frac{3\sqrt{3}}{16},-\frac{3\sqrt{3}}{16}\right)$

3 Variables

ALGORITHM

To find the maximum/minimum value of a differentiable function $f(x, y, z)$ subject to $g(x, y, z) = k$ such that $g \in C^1$, we evaluate $f(x, y, z)$ at all points (a, b, c) which satisfy one of the following:

(1) $\nabla f(a, b, c) = \lambda \nabla g(a, b, c)$ and $g(a, b, c) = k$.

(2) $\nabla g(a, b, c) = (0, 0, 0)$ and $g(a, b, c) = k$.

(3) (a, b, c) is an edge point of the surface $g(x, y, z) = k$. (See Remark below.)

The maximum/minimum value of $f(x, y, z)$ is the largest/smallest value of f obtained from all points found in (1)-(3).

Example 4

Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is closest to the point $(1, 2, 2)$.

$$\boxed{\begin{array}{ll} \min & f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-2)^2 \\ \text{s.t} & x^2+y^2+z^2 = 1 \end{array}}$$

→ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ $\quad g(x,y,z) = 1$

解
$$\boxed{\begin{array}{l} 2(x-1) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2(z-2) = 2\lambda z \\ x^2+y^2+z^2 = 1 \end{array}}$$

$$\frac{x-1}{x} = \frac{y-2}{y} = \frac{z-2}{z} \Rightarrow y = 2x \quad z = 2x. \quad y = z.$$

$\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ $\quad \left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$

→ check $\nabla g(x,y,z) = (0,0,0)$ $\quad g(x,y,z) = 1$

$\because \nabla g(x,y,z) = (0,0,0) \rightarrow x = y = z = 0$ doesn't satisfy constraints

$\therefore$ No points

→ check endpoints

$\because$ constrains is closed curve

$\therefore$ No endpoints

→ evaluate

$f(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) = 4 \qquad f(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}) = 16$

the point closest to point (1, 2, 2)

多个 variables

The method of Lagrange multipliers can be generalized to functions of n variables $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$ and with r constraints of the form

$$g_1(\vec{x}) = 0, \quad g_2(\vec{x}) = 0, \quad \ldots, \quad g_r(\vec{x}) = 0 \tag{*}$$

In order to find the possible maximum and minimum points of f subject to the constraints (*), we have to find all the points $\vec{a}$ such that

$$\nabla f(\vec{a}) = \lambda_1 \nabla g_1(\vec{a}) + \cdots + \lambda_r \nabla g_r(\vec{a}), \quad \text{and} \quad g_i(\vec{a}) = 0, \qquad 1 \leq i \leq r$$

The scalars $\lambda_1, \ldots, \lambda_r$ are the Lagrange multipliers. When $r = 1$, and $n = 2$ or 3, this reduces to the previous algorithms.

11.1 Polar Coordinates

– Coordinate System 坐标系

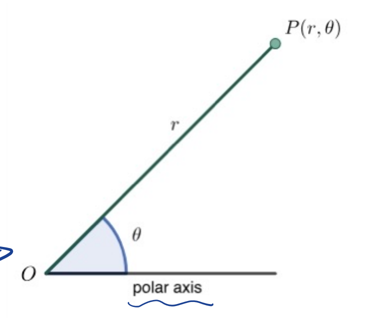
def. a system for representing the location of a point in a space by ordered n-tuple

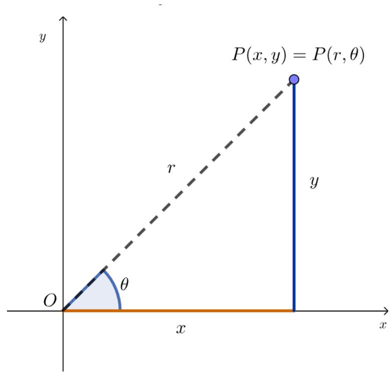
such as · Cartesian coordinate system.

- polar coordinates 极坐标
- cylindrical coordinates 圆柱坐标
- spherical coordinates 球坐标

– Polar Coordinates

pole →





- polar coordinates of P: $(r, \theta) = (r, \theta + 2\pi k)$ $k \in \mathbb{Z}$

  按 clockwise 算 (θ)

  non-negative (r)

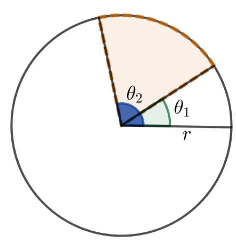
- equations

  $x = r\cos\theta$ $y = r\sin\theta$ $r = \sqrt{x^2+y^2}$ $\tan\theta = \frac{y}{x}$.

  $x^2+y^2=r^2$

  $0 \le \theta \le 2\pi$ 中, $\tan\theta = \frac{y}{x}$ 出现两次

  ☆ 得出 θ 前必须确认象限.



- area between θ_2 & θ_1

  $\theta_2 > \theta_1$ 橙色部分面积: $\text{Area} = \frac{\theta_2 - \theta_1}{2\pi} \pi r^2 = \frac{1}{2} r^2 (\theta_2 - \theta_1)$

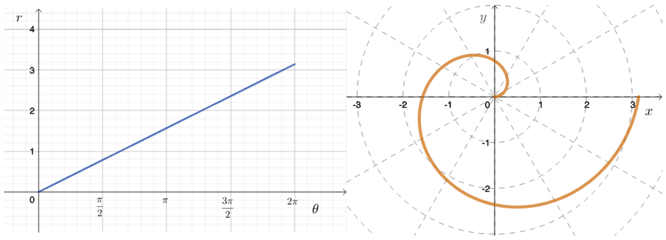
- area between curves

  步骤: 1. 找 2条 curves 的交点
  2. 画图. 分割.
  3. 积分

Example 5

Sketch the polar equation $r = \frac{1}{2}\theta,\ 0 \le \theta \le 2\pi$.

← 作图



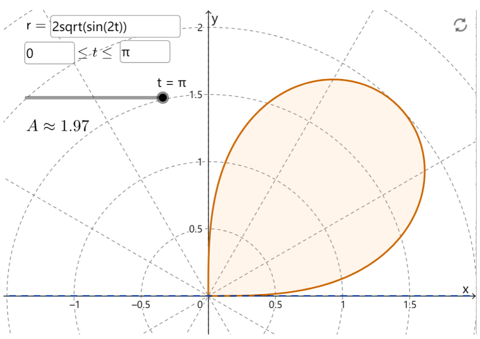
Remark

The polar equation $r = e^{\theta}$ gives a **logarithmic spiral** which often appears in nature. The nautilus shell is a nice example of this spiral.

FlamingPumpkin/E+/Getty Images

Your Turn 2

Find the area inside the curve $r = 2\sqrt{\sin 2\theta}$.



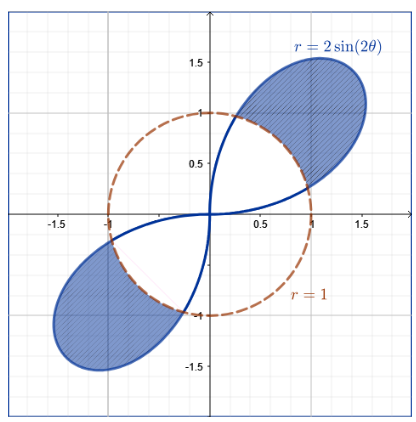
→ under $\sqrt{\ }$, $\therefore \sin 2\theta \ge 0$

↳ 只在一三象限

$$A = 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[2\sqrt{\sin(2\theta)}\right]^2 d\theta = 4$$

Example 11

Find the area inside $r = 2\sin(2\theta)$, but outside $r = 1$.



→ 找交点

$2\sin(2\theta) = 1.$ $\quad 2\theta = \frac{\pi}{6} \quad 2\theta = \frac{5\pi}{6}$

→ 画图. 分割

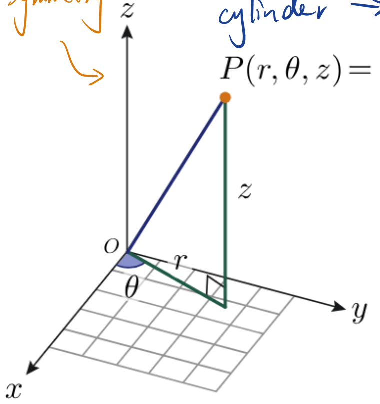
→ 积分

$$A = 2\left(\int_{\frac{\pi}{12}}^{\frac{5}{12}\pi} \frac{1}{2}(2\sin(2\theta))^2\, d\theta - \int_{\frac{\pi}{12}}^{\frac{5}{12}\pi} \frac{1}{2}\cdot 1^2\, d\theta\right)$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

11.2 Cylindrical Coordinates 三维

axis of symmetry



cylinder → Cartesian

$P(r, \theta, z) = P(x, y, z)$

$\theta = \frac{y}{z}$

$$\boxed{P(x,y,z) = P(r\cos\theta, r\sin\theta, z)}$$

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = z \qquad z = z$$

$$x^2 + y^2 = r^2$$

Example 1

Convert the following points from cylindrical coordinates to Cartesian coordinates and verify the results using the app above:

a. $(2, 0, 0)$

b. $(0, \pi, 2)$

a. $(2,0,0) \rightarrow r=2 \quad \theta=0 \quad z=0$

$x = 2\cos(0) = 2 \quad y = 2\sin(0) = 0 \quad z=0 \qquad (2,0,0)$

b. $(0,\pi,2) \rightarrow r=0 \quad \theta=\pi \quad z=2$

$x = 0\cos\pi = 0 \quad y = 0 \quad z=2 \qquad (0,0,2)$

Example 2

Convert the following points from Cartesian coordinates to cylindrical coordinates and verify the results using the app above:

a. $(1, 1, 3)$

b. $(1, -\sqrt{3}, 1)$

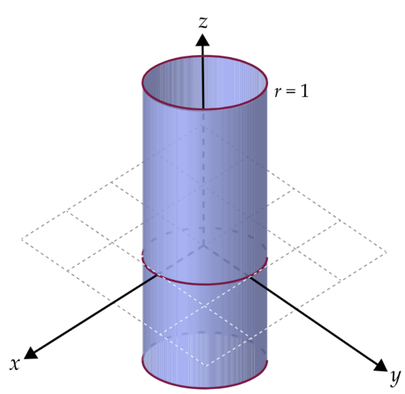
a. $r = \sqrt{1^2+1^2} = \sqrt{2} \quad \tan\theta = 1 \rightarrow \theta = \frac{\pi}{4} \qquad (\sqrt{2}, \frac{\pi}{4}, 3)$

b. $r = \sqrt{1^2+(-\sqrt{3})^2} = 2 \quad \tan\theta = -\sqrt{3}. \rightarrow \theta = \frac{5\pi}{3} \qquad (2, \frac{5}{3}\pi, 1)$

Example 3

Sketch the graph of $r = 1$ in cylindrical coordinates.

对于所有z. 都存在半径为1的圆.



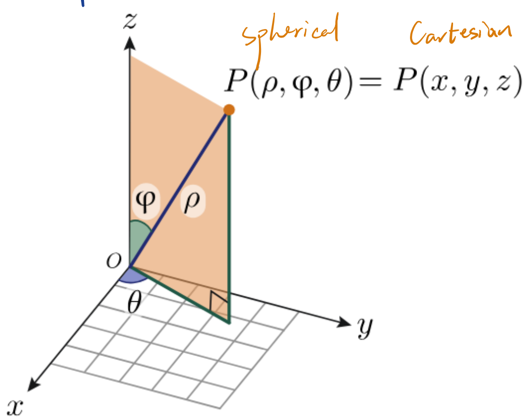
Example 4

Convert the equation $z = r^2\cos\theta$ to Cartesian coordinates.

$z = r^2\cos\theta = \sqrt{x^2+y^2} \cdot x = x\sqrt{x^2+y^2}$

11.3 Spherical Coordinates

- def. 3-demensional coordinate system.



$$\boxed{P(x, y, z) = P(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)}$$

$$x = \rho\sin\varphi\cos\theta \qquad \rho = \sqrt{x^2+y^2+z^2}$$

$$y = \rho\sin\varphi\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = \rho\cos\varphi \qquad \cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$x^2+y^2+z^2 = \rho^2$$

Example 1

Convert the following points from spherical coordinates to Cartesian coordinates.

a. $\left(1, \frac{\pi}{4}, \frac{\pi}{4}\right)$

b. $\left(1, \frac{\pi}{4}, \frac{5\pi}{4}\right)$

b. $x = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{5}{4}\pi\right) = -\frac{1}{2}$

$y = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{5}{4}\pi\right) = -\frac{1}{2}$

$z = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Example 2

Convert the following points from Cartesian coordinates to spherical coordinates.

a. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{3}\right)$

b. $(-1, -1, -1)$

a. $\rho = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (\sqrt{3})^2} = 2.$

$\cos\varphi = \frac{\sqrt{3}}{2} \qquad \varphi = \frac{\pi}{6}$

$\because \theta$ is in 1st-quadrant. $x, y > 0 \quad \therefore \tan\theta = 1 \quad \theta = \frac{\pi}{4}$

$\left(2, \frac{\pi}{6}, \frac{\pi}{4}\right)$

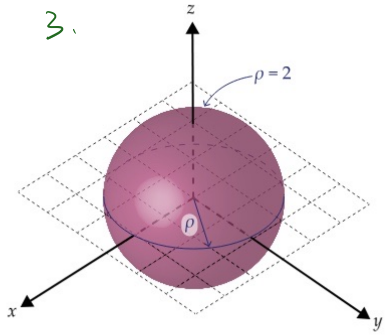
Example 3

Sketch $\rho = 2$.

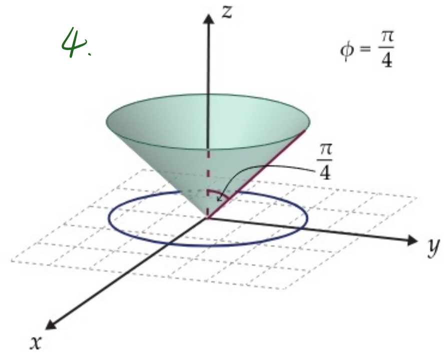
Example 4

Sketch $\varphi = \frac{\pi}{4}$.

3.



4.



Example 5

Convert $\rho = \sin\varphi\cos\theta$ to Cartesian coordinates.

$\rho^2 = \rho\sin\varphi\cos\theta.$

apply conversion equation. $x^2+y^2+z^2=x.$

$(x-\frac{1}{2})^2+y^2+z^2=\frac{1}{4}$

Example 6

Convert $z^2 = x^2 + y^2$ to spherical coordinates.

$\rho^2\cos^2\varphi = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta$

$\cos^2\varphi = \sin^2\varphi(\cos^2\theta+\sin^2\theta)$

$\tan^2\varphi = \pm 1. \to \underline{\varphi=\frac{\pi}{4}}$ $\underline{\varphi = \frac{3}{4}\pi.}$

圆锥 centred at positive z-axis

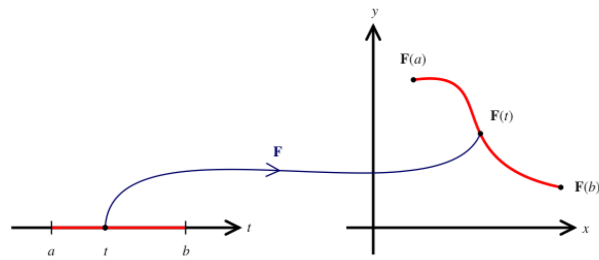
negative

12.1 The Geometry of Mapping

DEFINITION
Vector-Valued Function

A function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$ is called a **vector-valued** function.

$(x, y) = F(t) = (f(t), g(t))$



DEFINITION
Mapping

A vector-valued function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is a subset of $\mathbb{R}^n$ is called a **mapping** (or transformation).

$u = f(x, y) \qquad v = g(x, y)$

$(u, v) = F(x, y) = (f(x, y), g(x, y))$

component functions of mapping

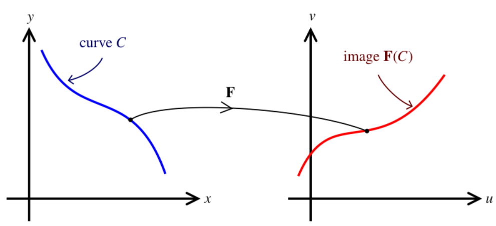


image of C under F

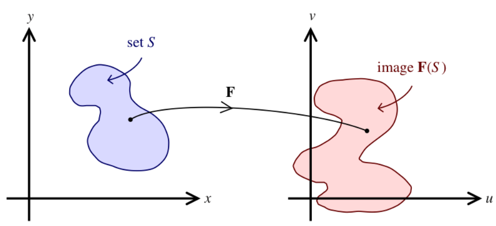


image of S under F

Example 1

Consider the mapping defined by $(u, v) = F(x, y) = \left(\underbrace{\frac{1}{2}(x+y)}_{}^{u}, \underbrace{\frac{1}{2}(-x+y)}_{}^{v}\right)$.

a. Find the images of the lines $x = k$ and $y = \ell$ under F.

$u = \frac{1}{2}(x+y) \qquad v = \frac{1}{2}(-x+y)$

convert $x = k$, $y = \ell$ in terms of u & v.

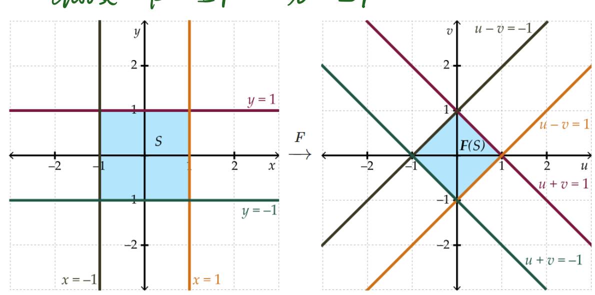
$x = u - v \qquad y = u + v.$

$x = k \xrightarrow{\text{mapping}} u - v = k$

$y = \ell \xrightarrow{\text{mapping}} u + v = \ell$

b. Find the image of the square $S = \{(x, y) \mid |x| \le 1, |y| \le 1\}$ under F.

choose $k = \pm 1 \qquad \ell = \pm 1$



Example 2

Find the image of $D = \{(x, y) \mid -1 \le x \le 3, 0 \le y \le 2\}$ under the mapping

$$(u, v) = T(x, y) = (x^2 - y^2, xy)$$

→ For the line $x = -1$. $0 \le y \le 2$.

we get $u = (-1)^2 - y^2 = 1 - y^2$
$v = (-1)y = -y$

equation in u-v plane $u = 1 - (-v)^2 = 1 - v^2$

$\because v = -y$. $0 \le y \le 2$.

$\therefore 0 \le -v \le 2$. $\Rightarrow -2 \le v \le 0$

→ For line $x = 3$, $0 \le y \le 2$.

we get $v = 3y$

equation in u-v plane $u = 3^2 - y^2 = 9 - y^2 = 9 - (\frac{1}{3}v)^2 = 9 - \frac{1}{9}v^2$

$0 \le \frac{1}{3}v \le 2 \Rightarrow 0 \le v \le 6$.

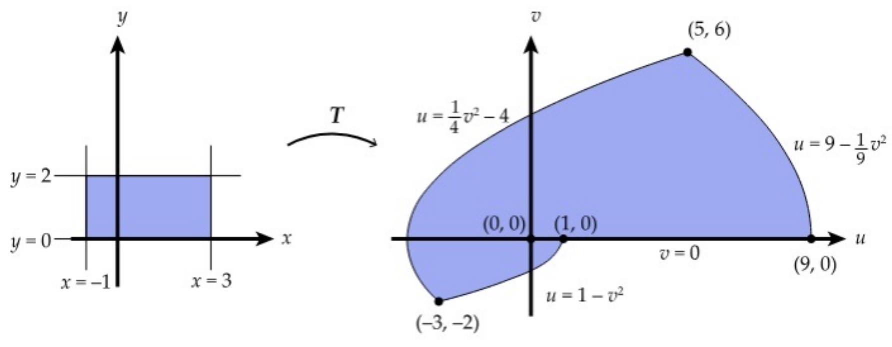
→ For line $y = 2$, $-1 \le x \le 3$.

we get $v = 2x$.

equation in u-v plane $u = x^2 - 2^2 = \frac{1}{4}v^2 - 4$. $-2 \le v \le 6$

→ For line $y = 0$, $-1 \le x \le 3$.

we get $v = 0$. $u = x^2 - 0^2 = x^2$.



12.2 Linear approximation of a mapping

DEFINITION
Derivative Matrix

The **derivative matrix** of a mapping defined by

$$F(x,y) = (f(x,y), g(x,y))$$

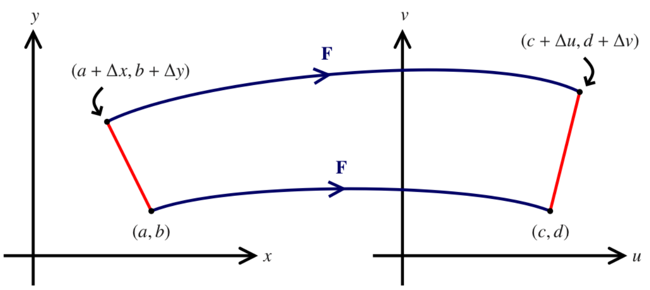
is denoted DF and defined by

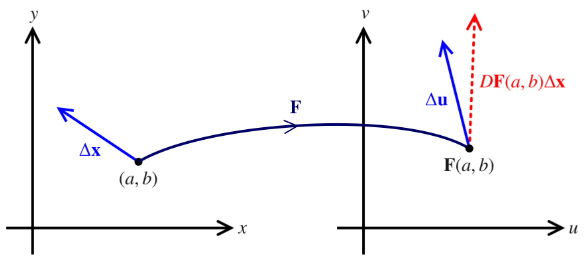
$$DF = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$\Delta u = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$ $\quad \Delta x = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$

increment form of lin-approx. for mapping : $\Delta u \approx DF(a,b)\Delta x$

$\Delta x \to 0$. lin-approx for mapping : $F(x,y) \approx F(a,b) + DF(a,b)\Delta x$





Example 1

Find the derivative matrix of the mapping

$$(u,v) = F(x,y) = (\underbrace{x^2 \sin y}_{f(x)}, \underbrace{y^2 \cos x}_{g(x)})$$

$f(x,y) = x^2 \sin y \quad g(x,y) = y^2 \cos x.$

$$DF(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x\sin y & x^2 \cos y \\ -y^2 \sin x & 2y\cos x \end{bmatrix}$$

Example 2

Consider the mapping defined by

$$(u, v) = F(x, y) = \left(-x + \sqrt{x^2 + y^2}, x + \sqrt{x^2 + y^2}\right)$$

as in the previous Your Turn exercise. Use the linear approximation to estimate the image of the point $(3.02, 3.99)$ under F.

$$DF(x, y) = \begin{bmatrix} -1 + \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ 1 + \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$$DF(3, 4) = \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5} \end{bmatrix}$$

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx DF(3, 4) \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 0.02 \\ -0.01 \end{bmatrix} = \begin{bmatrix} -0.016 \\ 0.024 \end{bmatrix}$$

$$F(3.02, 3.99) \approx (2, 8) + (-0.016, 0.024) = (1.984, 8.024)$$

Generalization

A mapping F from $\mathbb{R}^n$ to $\mathbb{R}^m$ is defined by a set of m component functions:

$$\begin{aligned} u_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ u_m &= f_m(x_1, \ldots, x_n) \end{aligned}$$

Or, in vector notation

$$\mathbf{u} = F(\mathbf{x}) = (f_1(\mathbf{x}), \cdots f_m(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^n$$

If we assume that F has continuous partial derivatives, then the derivative matrix of F is the $m \times n$ matrix defined by

$$DF(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

As expected, the linear approximation for F at $\mathbf{a}$ is

$$F(\mathbf{x}) \approx F(\mathbf{a}) + DF(\mathbf{a})\Delta\mathbf{x}$$

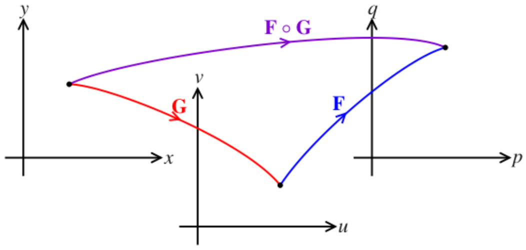
where

$$\Delta\mathbf{u} = \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_m \end{bmatrix} \in \mathbb{R}^m, \qquad \Delta\mathbf{x} = \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} \in \mathbb{R}^n$$

12.3 Composite mapping and Chain Rule

Consider successive mappings F and G of $\mathbb{R}^2$ into $\mathbb{R}^2$, defined by

$$F : \begin{cases} p = p(u, v) \\ q = q(u, v) \end{cases} \qquad G : \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases} \tag{12.1}$$



The composite mapping $F \circ G$, defined by

$$\begin{cases} p = p\big(u(x, y), v(x, y)\big) \\ q = q\big(u(x, y), v(x, y)\big) \end{cases} \tag{12.2}$$

maps the xy-plane directly into the pq-plane.

THEOREM 1 **(Chain Rule in Matrix Form)**

Let F and G be mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. If G has continuous partial derivatives at (x, y) and F has continuous partial derivatives at $(u, v) = G(x, y)$, then the composite mapping $F \circ G$ has continuous partial derivatives at (x, y) and

$$D(F \circ G)(x, y) = DF(u, v)DG(x, y)$$

Example 1

Consider the mappings G and F defined by

$$(u, v) = G(x, y) = (xy, x + y)$$
$$(p, q) = F(u, v) = (u - v, u^2)$$

a. Find the composite mapping $F \circ G$

b. Find the derivative matrices DG, DF, $D(F \circ G)$, and verify the Chain Rule formula.

a. $(p, q) = F(G(x, y)) = F(xy, x+y) = (xy - x - y, x^2y^2)$

b. $DG(x, y) = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$ $\quad DF(u, v) = \begin{bmatrix} 1 & -1 \\ 2u & 0 \end{bmatrix}$ $\quad D(F \circ G)(x, y) = \begin{bmatrix} y-1 & x-1 \\ 2xy^2 & 2x^2y \end{bmatrix}$

$$\begin{aligned} DF(u,v)\,DG(x,y) &= \begin{bmatrix} 1 & -1 \\ 2u & 0 \end{bmatrix}\begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} y-1 & x-1 \\ 2uy & 2ux \end{bmatrix} \\ &= \begin{bmatrix} y-1 & x-1 \\ 2xy^2 & 2x^2y \end{bmatrix} \ (\text{代入 } u = xy) \\ &= D(F \circ G)(x, y) \end{aligned}$$

13.1 The inverse mapping theorem

- Invertible mapping

DEFINITION

Invertible Mapping

Inverse Mapping

Let F be a mapping from a set D_{xy} onto a set D_{uv}. If there exists a mapping F^{-1}, called the **inverse of** F which maps D_{uv} onto D_{xy} such that

$$(x, y) = F^{-1}(u, v) \quad \text{if and only if} \quad (u, v) = F(x, y)$$

then F is said to be **invertible** on D_{xy}.

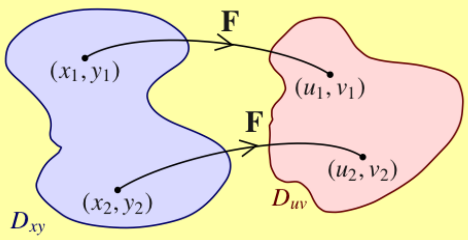
$$(F^{-1} \circ F)(x, y) = (F \circ F^{-1})(x, y) = (x, y) \qquad (*)$$

- One-to-one

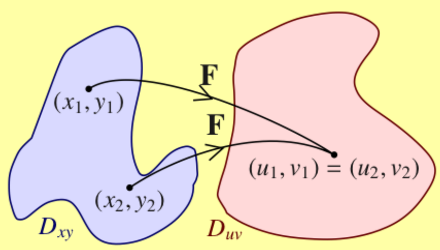
DEFINITION

One-to-One

A mapping F from $\mathbb{R}^2$ to $\mathbb{R}^2$ is said to be **one-to-one** on a set D_{xy} if and only if $F(a, b) = F(c, d)$ implies $(a, b) = (c, d)$, for all $(a, b), (c, d) \in D_{xy}$.



F is one-to-one



F is not one-to-one

- One-to-one ⇒ invertible

THEOREM 1 Let F be a mapping from a set D_{xy} onto a set D_{uv}. If F is one-to-one on D_{xy}, then F is invertible on D_{xy}.

- inverse of derivative matrix

THEOREM 2 Consider a mapping F which maps D_{xy} onto D_{uv}. If F has continuous partial derivatives at $\mathbf{x} \in D_{xy}$ and there exists an inverse mapping F^{-1} of F which has continuous partial derivatives at $\mathbf{u} = F(\mathbf{x}) \in D_{uv}$, then

$$DF^{-1}(\mathbf{u})DF(\mathbf{x}) = I$$

proof: By Chain Rule, $DF^{-1}(u)DF(x) = D(F^{-1} \circ F)(x)$

By $(*)$, $D(F^{-1} \circ F)(x) = Dx = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

EXAMPLE 1 Consider the mapping defined by

$$(u, v) = F(x, y) = (y + x^2, x)$$

Solve for the inverse mapping F^{-1}. Find the derivative matrices DF and DF^{-1} and verify that $DF^{-1}(u, v)$ is the matrix inverse of $DF(x, y)$.

→ 相当于解 $\begin{cases} u = y + x^2 \\ v = x \end{cases}$ ⇒ $\begin{cases} x = v \\ y = u - v^2 \end{cases}$

∴ inverse mapping is $(x, y) = F^{-1}(u, v) = (v, u - v^2)$

→ derivative matrices:

$$DF(x,y) = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \qquad DF^{-1}(u,v) = \begin{bmatrix} 0 & 1 \\ 1 & -2v \end{bmatrix}$$

→ verify inverse.

$$DF^{-1}(u,v)\, DF(u,v) = \begin{bmatrix} 0 & 1 \\ 1 & -2v \end{bmatrix}\begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ (将 $v = x$ 代入)

– The Jacobian of a mapping

DEFINITION

Jacobian

The **Jacobian** of a mapping

$$(u, v) = F(x, y) = (u(x, y), v(x, y))$$

is denoted $\dfrac{\partial(u, v)}{\partial(x, y)}$, and is defined by

$$\frac{\partial(u, v)}{\partial(x, y)} = \det[DF(x, y)] = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

EXERCISE 1 Calculate the Jacobian $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ of the mapping F given by

$$(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)$$

→ find derivative matrix $DF(r, \theta)$

$$DF(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

→ calculate Jacobian

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$$

COROLLARY 3

Consider a mapping defined by

$$(u, v) = F(x, y) = (f(x, y), g(x, y))$$

which maps a subset D_{xy} onto a subset D_{uv}. Suppose that f and g have continuous partials on D_{xy}. If F has an inverse mapping F^{-1}, with continuous partials on D_{uv}, then the Jacobian of F is non-zero:

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0, \quad \text{on } D_{xy}$$

COROLLARY 4

(Inverse Property of the Jacobian)

If the hypotheses of Theorem 2 hold, then

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$$

proof: $I = DF^{-1}(u,v)\, DF(x,y) I$ (By theorem 2)

$\det I = \det(DF^{-1}(u,v)\, DF(x,y))$

$1 = \det(DF^{-1}(u,v)\, DF(x,y))$

$\because DF(x,y)$ is invertible $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$

$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

EXAMPLE 2

Consider the mapping defined by

$$(u, v) = F(x, y) = (e^x \cos y, e^x \sin y)$$

Show that $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$ on $\mathbb{R}^2$, but that F^{-1} does not exist on $\mathbb{R}^2$.

proof: $\frac{\partial(u,v)}{\partial(x,y)} = e^{2x} > 0 \quad \forall (x,y) \in \mathbb{R}^2$

$\because F(0,0) = F(0, 2\pi) = (1, 0)$

$\therefore F$ is not one-to-one on $\mathbb{R}$.

$\therefore F^{-1}$ DNE on $\mathbb{R}^2$

- Inverse mapping theorem

THEOREM 5 **(Inverse Mapping Theorem)**

If a mapping $(u, v) = F(x, y)$ has continuous partial derivatives in some neighborhood of (a, b) and $\dfrac{\partial(u, v)}{\partial(x, y)} \neq 0$ at (a, b), then there is a neighborhood of (a, b) in which F has an inverse mapping $(x, y) = F^{-1}(u, v)$ which has continuous partial derivatives.

EXAMPLE 3 Consider the mapping defined by

$$(u, v) = F(x, y) = (xy - x^2, x + y)$$

Show that F has an inverse mapping in a neighborhood of $(1, -2)$.

Jacobian of F : $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} y-2x & x \\ 1 & 1 \end{bmatrix} = y-3x$.

$\because (x,y)=(1,-2)$ Jacobian is non-zero.

$\therefore$ partial derivative of F are cts by cts theo.

$\therefore$ By inverse mapping theo, there is a neighbourhood of $(1,-2)$.

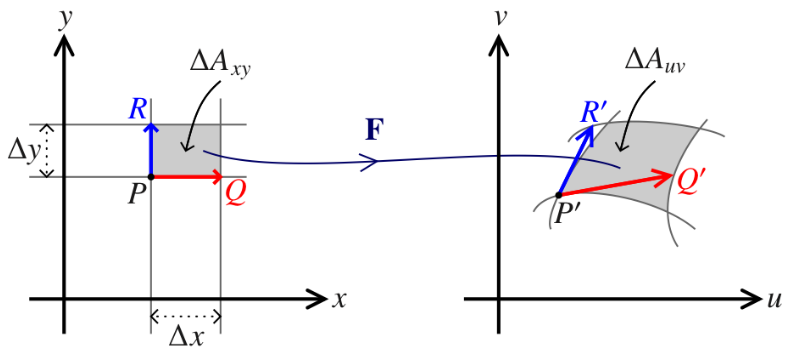
EXERCISE 2 Referring to Example 3, show that the inverse mapping is given by

$$(x, y) = F^{-1}(u, v) = \left(\frac{1}{4}(v + \sqrt{v^2 - 8u}), \frac{1}{4}(3v - \sqrt{v^2 - 8u})\right)$$

$$\begin{aligned}(F^{-1} \circ F)(x,y) &= F^{-1}(xy-x^2, x+y) \\ &= \left(\tfrac{1}{4}\left[x+y+\sqrt{(x+y)^2-8(xy-x^2)}\right], \tfrac{1}{4}\left[3(x+y)-\sqrt{(x+y)^2-8(xy-x^2)}\right]\right) \\ &= (x,y)\end{aligned}$$

13.2 Geometrical Interpretation of the Jacobian

2D



边 $\quad \vec{PQ} = \begin{bmatrix}\Delta x \\ 0\end{bmatrix} \qquad \vec{P'Q'} \approx \begin{bmatrix}u_x & u_y \\ v_x & v_y\end{bmatrix}\begin{bmatrix}\Delta x \\ 0\end{bmatrix} = \begin{bmatrix}u_x \Delta x \\ v_x \Delta x\end{bmatrix}$

$\vec{PR} = \begin{bmatrix}0 \\ \Delta y\end{bmatrix} \qquad \vec{P'R'} \approx \begin{bmatrix}u_x & u_y \\ v_x & v_y\end{bmatrix}\begin{bmatrix}0 \\ \Delta y\end{bmatrix} = \begin{bmatrix}u_y \Delta y \\ v_y \Delta y\end{bmatrix}$

面积 $\quad \Delta A_{xy} = \Delta x \Delta y \qquad \Delta A_{uv} \approx \left|\det \begin{bmatrix}u_x \Delta x & u_y \Delta y \\ v_x \Delta x & v_y \Delta y\end{bmatrix}\right|$

$$= \left|\det \begin{bmatrix}u_x & u_y \\ v_x & v_y\end{bmatrix}\right| \Delta x \, \Delta y$$

basic relation $\longrightarrow$ $= \left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| \Delta A_{xy}$ (Since $\Delta x, \Delta y > 0$)

Jacobian of mapping F 描述图像变换的情况

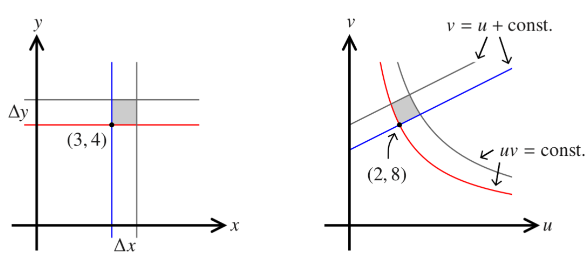
EXAMPLE 1 Calculate the approximate area of the image of a small rectangle of area $\Delta x \Delta y$, located at the point $(3, 4)$, under the mapping F defined by

$$(u, v) = F(x, y) = \left(-x + \sqrt{x^2 + y^2}, \quad x + \sqrt{x^2 + y^2}\right)$$

$\to$ $DF(3,4) = \begin{bmatrix}-\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5}\end{bmatrix}$

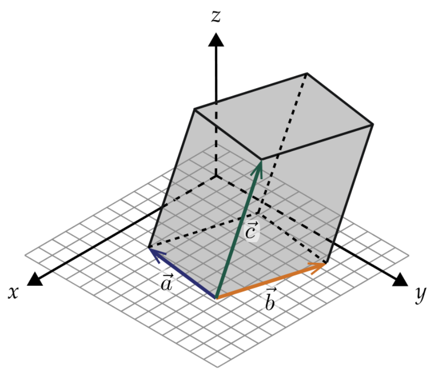
$\to$ Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix}-\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5}\end{bmatrix} = -\frac{8}{5}$

$\to$ Area $\quad \Delta A_{ux} \approx \frac{8}{5} \Delta A_{xy}$



3D

The volume of a parallelepiped which is defined by three vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is given by



$$\text{Volume} = \left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right|$$

Your Turn

Determine the scaling factor of the image of a small rectangular block in xyz-space under the mapping

$$(u, v, w) = F(x, y, z) = (xy, x + z, x^2 - yz^2)$$

$\Delta V_{uvw} \approx |a| \Delta V_{xyz}$, where $a =$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \det \begin{bmatrix} y & x & 0 \\ 1 & 0 & 1 \\ 2x & -z^2 & -2yz \end{bmatrix}$$

$$= 2x^2 + 2xyz + yz^3$$

$$\Delta V_{uvw} \approx |2x^2 + 2xyz + yz^2| \Delta V_{xyz}$$

nD

Definition: The Jacobian — General Form

For a mapping defined by

$$\vec{u} = F(\vec{x}) = \big(f_1(\vec{x}), \ldots, f_n(\vec{x})\big)$$

where $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{x} = (x_1, \ldots, x_n)$, the **Jacobian of F** is

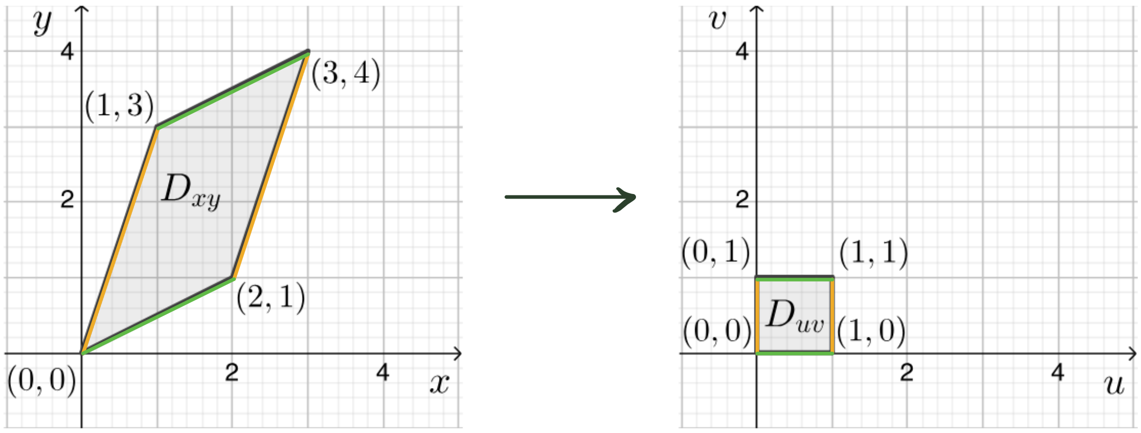
$$\frac{\partial(u_1, \ldots, u_n)}{\partial(x_1, \ldots, x_n)} = \det[DF(\vec{x})] = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

13.3 Constructing mappings

Q.

① Find a mapping F which transforms the parallelogram, D_{xy}, with vertices $(0,0)$, $(2,1)$, $(3,4)$, and $(1,3)$ in the xy-plane into the unit square, D_{uv}, $0 \le u \le 1$, $0 \le v \le 1$ in the uv-plane.

② Calculate the Jacobian of F and hence find the area of the parallelogram in the xy-plane.



① → boundary lines

$\boxed{\begin{array}{c} 2y-x=0 \\ 2y-x=5 \end{array}}$ $\boxed{\begin{array}{c} 3x-y=0 \\ 3x-y=5 \end{array}}$ $\boxed{\begin{array}{c} u=0 \\ u=1 \end{array}}$ $\boxed{\begin{array}{c} v=0 \\ v=1 \end{array}}$

→ first pair $u=0$ $u=1$ $u=\frac{2y-x}{5}$

second pair $v=0$ $v=1$ $v=\frac{3x-y}{5}$

$$(u,v)=F(x,y)=\left(\frac{2y-x}{5}, \frac{3x-y}{5}\right)$$

② → Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix} = -\frac{1}{5}$$

→ ∵ mapping is linear

∴ relation: $A_{uv} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| A_{xy} = \frac{1}{5}A_{xy}$

So $A_{\square} = 5$ square units

14.1 Double integral

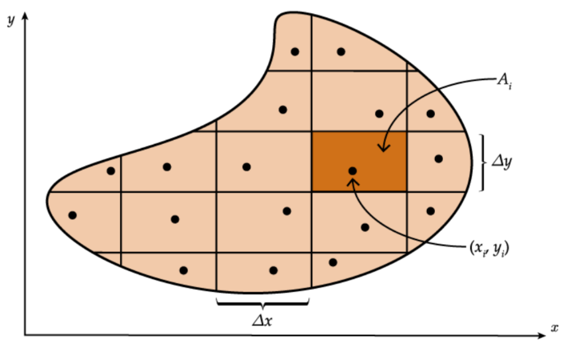
DEFINITION
Integrable

Let $D \subset \mathbb{R}^2$ be closed and bounded. Let P be a partition of D as described above, and let $|\Delta P|$ denote the length of the longest side of all rectangles in the partition P. A function $f(x, y)$ which is bounded on D is **integrable** on D if all Riemann sums approach the same value as $|\Delta P| \to 0$.

DEFINITION
Double Integral

If $f(x, y)$ is integrable on a closed bounded set D, then we define the **double integral** of f on D as

$$\iint_D f(x,y)\,dA = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$



相当于取无数小份，将面积相加

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i = \sum_{i=1}^{n} f(x_i, y_i)\Delta x_i \Delta y_i$$

- Double integral as areas

Specialize f to be constant function with value 1: $f(x,y)=1$. $\forall (x,y) \in D$

area of all rectangle in P : $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$

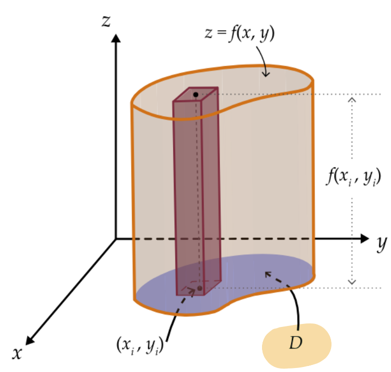
$$A(D) = \iint_D 1\,dA$$

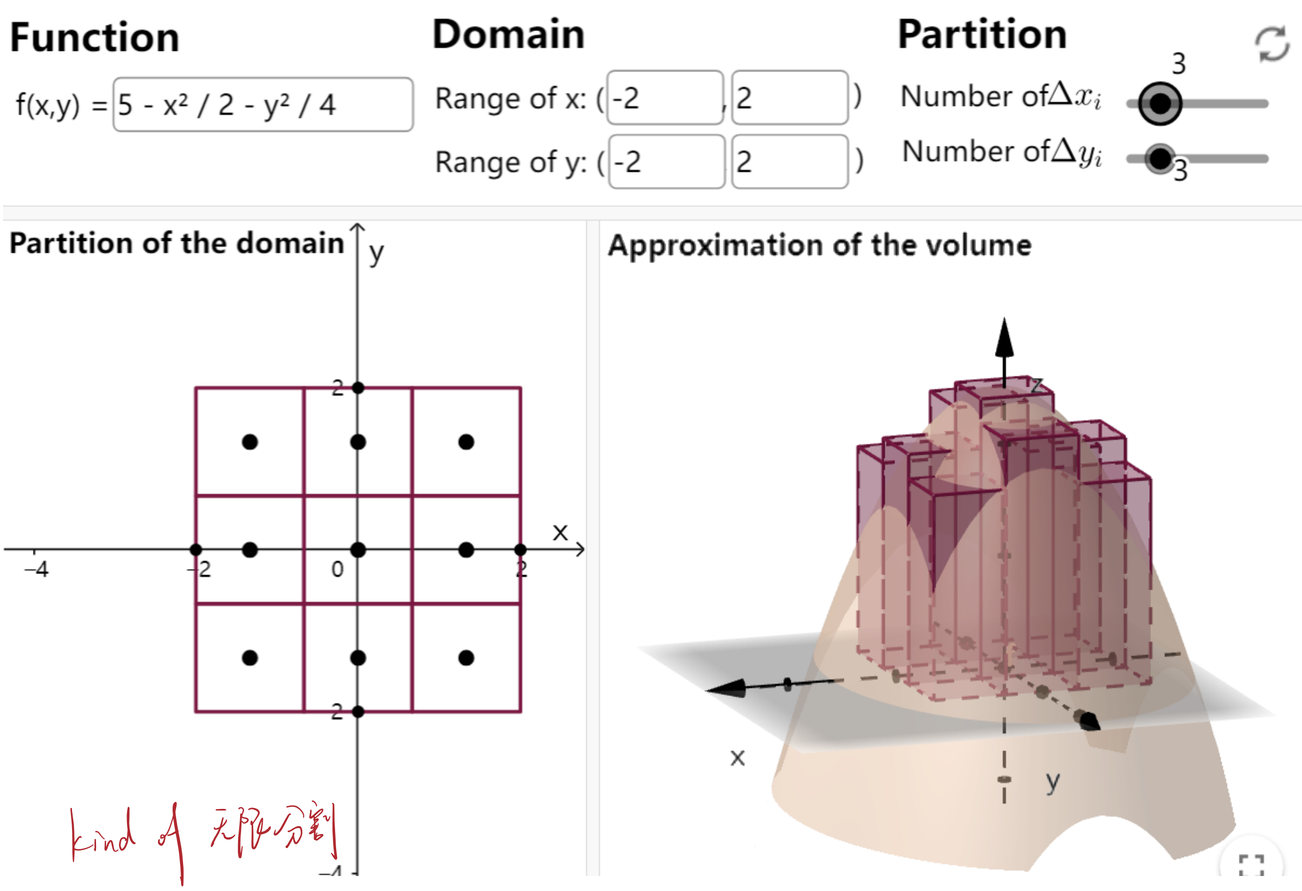
- Double integral as Volume

$$S = \{(x, y, z) \mid 0 < z < f(x,y),\ (x,y) \in D\}$$

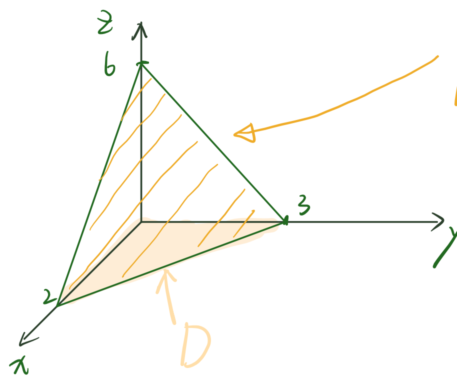
$$V(S) \approx \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i$$

$$V(S) = \iint_D f(x,y)\,dA.$$



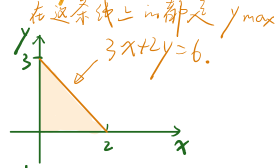


Q. Find the vol of the solid in the 1st octant (三维的象限) of $\mathbb{R}^3$ $(x, y, z \geq 0)$



plane $3x+2y+z=6.$ $\Rightarrow$ $z=6-3x-2y$

$V=\iint_D 6-3x-2y \, dA$, where D is

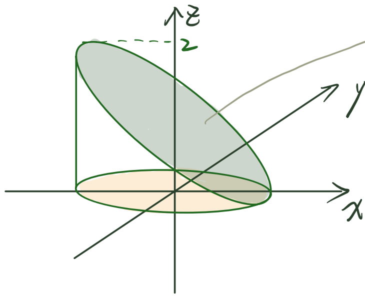


$= \int_{x\,min=0}^{x\,max=2} \left(\int_{y\,min(x)=0}^{y\,max(x)=3-\frac{3}{2}x} 6-3x-2y \, dy \right) dx$

$= \int_0^2 \left(\int_0^{3-\frac{3}{2}x} 6-3x-2y \, dy \right) dx$

$=6$

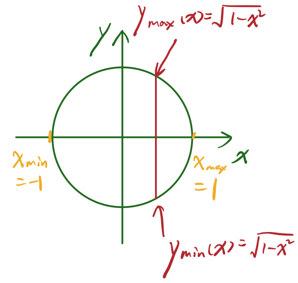
Q. Find the volumn of the solid inside the cylinder $x^2+y^2=1$, above xy-plane and below the plane $z+x=1$



$$Vol = \iint_D 1-x \, dA \text{, where } D \text{ is}$$

$$= \int_{x\,min}^{x\,max} \left(\int_{y\,min(x)}^{y\,max(x)} 1-x \, dy \right) dx$$

$$= \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1-x \, dy \right) dx = \pi$$



$$Vol = \iint_D 1-x \, dA.$$

$$= \iint_D 1 \, dA - \iint_D x \, dA$$

$$= \pi$$

- average value

Average Value of a Function

The double integral is also used to define the average value of a function $f(x, y)$ over a set $D \subset \mathbb{R}^2$. Recall for a function of one variable, $f(x)$, the average value of f over an interval $[a, b]$, denoted f_{av}, is defined by

$$\boxed{f_{av} = \frac{1}{b-a} \int_a^b f(x) \, dx}$$

Similarly, for a function of two variables $f(x, y)$, we can define the average value of f over a closed and bounded subset D of $\mathbb{R}^2$ by

$$\boxed{f_{av} = \frac{1}{A(D)} \iint_D f(x, y) \, dA}$$

Suppose $f(x,y)$ is cts on D. What is f_{avg}?

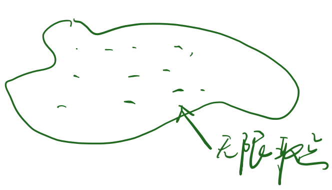
$$f_{avg} \approx \frac{f(x_1, y_1) + \cdots + f(x_n, y_m)}{\# \text{ pts}}$$

$$= \frac{\Sigma f(x_i, y_j) \Delta A}{\Sigma 1 \quad \Delta A}$$

$$= \lim_{\Delta A \to 1} \left(\frac{\Sigma f(x_i, y_j)}{\Sigma 1} \right)$$

$$\rightarrow \quad = \frac{\iint_D f(x,y) \, dA}{\iint_D 1 \, dA}$$

$$= \frac{1}{\text{area}(D)} \iint_D f(x,y) \, dA.$$



- Properties of double integral

THEOREM 1 **(Linearity)**

If $D \subset \mathbb{R}^2$ is a closed and bounded set and f and g are two integrable functions on D, then for any constant c:

$$\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA$$

$$\iint_D cf\,dA = c\iint_D f\,dA$$

THEOREM 2 **(Basic Inequality)**

If $D \subset \mathbb{R}^2$ is a closed and bounded set and f and g are two integrable functions on D such that $f(x, y) \leq g(x, y)$ for all $(x, y) \in D$, then

$$\iint_D f\,dA \leq \iint_D g\,dA$$

用于获得无法精确算出来的 double integral 的值

THEOREM 3 **(Absolute Value Inequality)**

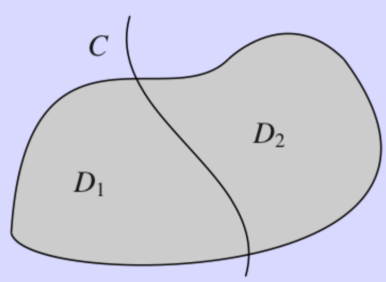
If $D \subset \mathbb{R}^2$ is a closed and bounded set and f is an integrable function on D, then

$$\left|\iint_D f\,dA\right| \leq \iint_D |f|\,dA$$

THEOREM 4 **(Decomposition)**

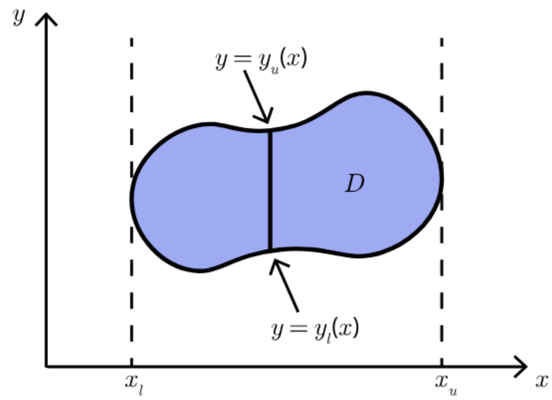
Assume $D \subset \mathbb{R}^2$ is a closed and bounded set and f is an integrable function on D. If D is decomposed into two closed and bounded subsets D_1 and D_2 by a piecewise smooth curve C, then

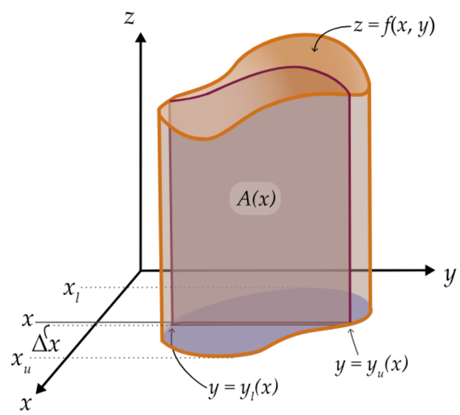
$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$



$D = D_1 \cup D_2$

14.2 Interated integrals





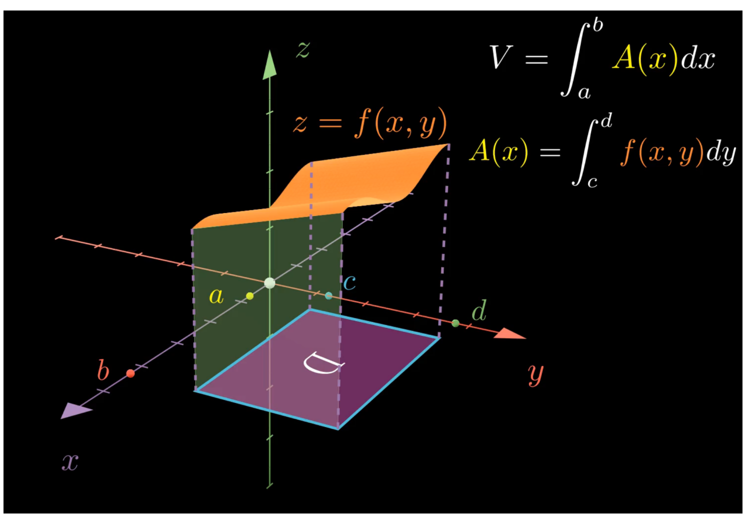
$$A(x) = \int_{y_l(x)}^{y_u(x)} f(x,y)\,dy \longrightarrow V = \int_{x_l}^{x_u} \left(\int_{y_l(x)}^{y_u(x)} f(x,y)\,dy\right) dx.$$

THEOREM 1 Let D be the subset of $\mathbb{R}^3$ defined by

$$z_\ell(x,y) \le z \le z_u(x,y) \quad \text{and} \quad (x,y) \in D_{xy}$$

where z_ℓ and z_u are continuous functions on D_{xy}, and D_{xy} is a closed bounded subset in $\mathbb{R}^2$, whose boundary is a piecewise smooth closed curve. If $f(x,y,z)$ is continuous on D, then

$$\iiint_D f(x,y,z)\,dV = \iint_{D_{xy}} \int_{z_\ell(x,y)}^{z_u(x,y)} f(x,y,z)\,dz\,dA$$



$$V = \int_a^b A(x)dx$$

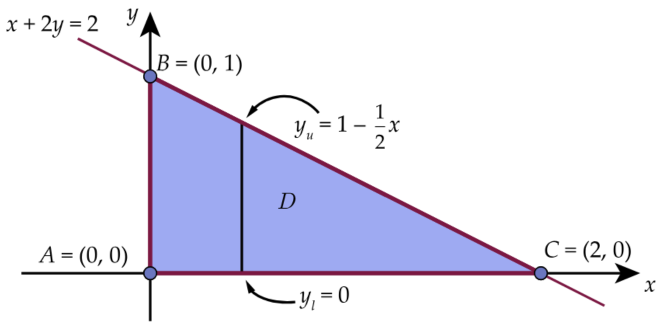
$$A(x) = \int_c^d f(x,y)dy$$

$$V = \int_a^b \left(\int_c^d f(x,y)dy\right) dx$$

Q.

Evaluate $\displaystyle\iint_D xy\,dA$ where D is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.

→ Sketch the region D.



→ Set up inequalities

bounded above $y_l = 0$

bounded below $x+2y=2$

x in 范围 $0 \le x \le 2$ $0 \le y \le 1-\frac{1}{2}x$

→ Set up & evaluate integral

$$V = \iint_D xy\,dA = \int_{x=0}^{2}\underline{\int_{y=0}^{1-\frac{1}{2}x} xy\,dy}\,dx$$

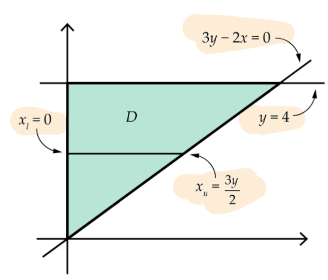
$$= \frac{1}{2}\int_0^2 x\left(1-\frac{1}{2}x\right)^2 dx$$

$$= \frac{1}{6}$$

Example 5

Find the volume of the solid S in the first octant ($x \ge 0$, $y \ge 0$, $z \ge 0$) bounded by the cylinder $y^2 + z^2 = 16$, and the planes $3y - 2x = 0$, $x = 0$, $z = 0$.

$$0 \le z \le \sqrt{16-y^2}$$



$$V = \iint_D \sqrt{16-y^2}\,dA$$

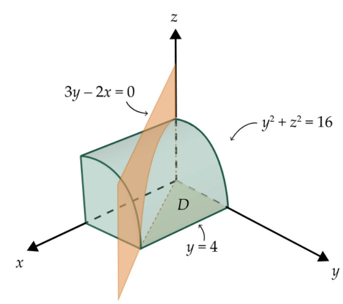
$$= \int_0^4 \int_0^{\frac{3}{2}y} \sqrt{16-y^2}\,dx\,dy$$

$$= \int_0^4 \left[\sqrt{16-y^2}\,(x)\right]_0^{\frac{3}{2}y} dy$$

$$= \int_0^4 \frac{3}{2}y\sqrt{16-y^2}\,dy$$

$$= \left[-\frac{1}{2}(16-y^2)^{\frac{3}{2}}\right]_0^4$$

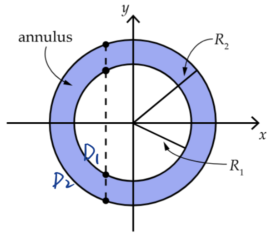
$$= 32$$



- Decomposition theorem.

对于较为复杂的区域，无法用不等式来描述.

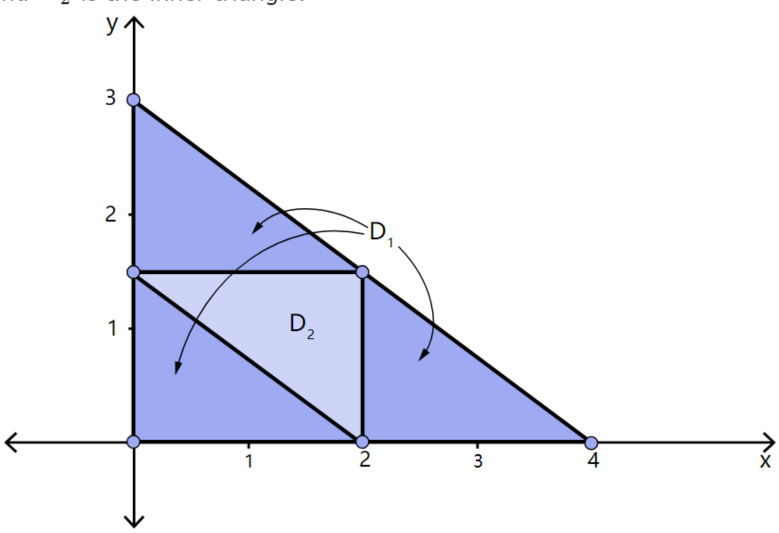
ep.



$$\iint_D f(x,y)\,dA = \iint_{D_2} f(x,y)\,dA - \iint_{D_1} f(x,y)\,dA$$

Example 6

Evaluate the integral $\iint_D (x^2+y^2)dA$ where D is the region illustrated below. $D = D_1 - D_2$ where D_1 is the outer triangle and D_2 is the inner triangle.



D_1 $\quad 0 \le x \le 4 \qquad 0 \le y \le -\frac{3}{4}x+3$

D_2 $\quad 0 \le x \le 2 \qquad -\frac{3}{4}x+\frac{3}{2} \le y \le \frac{3}{2}$

$$\iint_D (x^2+y^2)\,dA = \iint_{D_1} (x^2+y^2)\,dA - \iint_{D_2} (x^2+y^2)\,dA$$

$$= \int_0^4 \int_0^{-\frac{3}{4}x+3} (x^2+y^2)\,dy\,dx - \int_0^2 \int_{-\frac{3}{4}x+\frac{3}{2}}^{\frac{3}{2}} (x^2+y^2)\,dy\,dx$$

$$= 25 - \frac{25}{16}$$

$$= \frac{375}{16}$$

14.3 Change of variables

THEOREM 1 **(Change of Variables Theorem)**

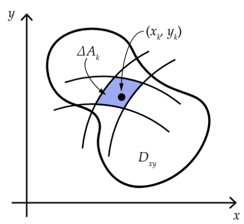
Let each of D_{uv} and D_{xy} be a closed bounded set whose boundary is a piecewise-smooth closed curve. Let

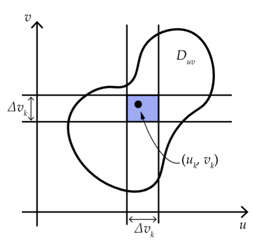
$$(x, y) = G(u, v) = (g(u, v), h(u, v))$$

be a one-to-one mapping of D_{uv} onto D_{xy}, with $g, h \in C^1$, and $\dfrac{\partial(x, y)}{\partial(u, v)} \neq 0$ except for possibly on a finite collection of piecewise-smooth curves in D_{uv}. If $f(x, y)$ is continuous on D_{xy}, then

$$\iint_{D_{xy}} f(x, y)\, dx\, dy = \iint_{D_{uv}} f\big(g(u, v), h(u, v)\big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

$\varphi : T \to D \qquad f : D \to \mathbb{R}. \qquad \int_D f(\vec{x})\, d^n\vec{x} = \int_T f(\varphi(\vec{u}))\, |\det D\varphi(\vec{u})|\, d^n\vec{u}$





$$\iint_{D_{xy}} G(x, y)\, dx\, dy = \lim_{\Delta P^* \to 0} \sum_{i=1}^{n} G(x_i, y_i)\Delta A_i$$

$$= \lim_{\Delta P^* \to 0} \sum_{i=1}^{n} G\Big(f(u_i, v_i), g(u_i, v_i)\Big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|_{(u_i, v_i)} \Delta A_i$$

$$= \iint_{D_{uv}} G\Big(f(u, v), g(u, v)\Big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA$$

Example 1

Evaluate $\displaystyle\iint_{D_{xy}} (x + y)\, dA$, where D_{xy} is the set bounded by the parallelogram with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, and $(3, 4)$.

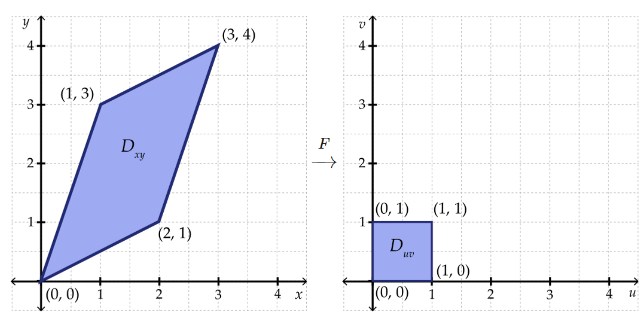
$(u, v) = F(x, y) = \left(\frac{1}{5}(2y - x), \frac{1}{5}(3x - y)\right)$

Jacobian of F $\quad \dfrac{\partial(u, v)}{\partial(x, y)} = -\dfrac{1}{5}$

$(x, y) = F^{-1}(u, v) = (u + 2v, 3u + v)$

Jacobian of F^{-1} $\quad \dfrac{\partial(x, y)}{\partial(u, v)} = -5$

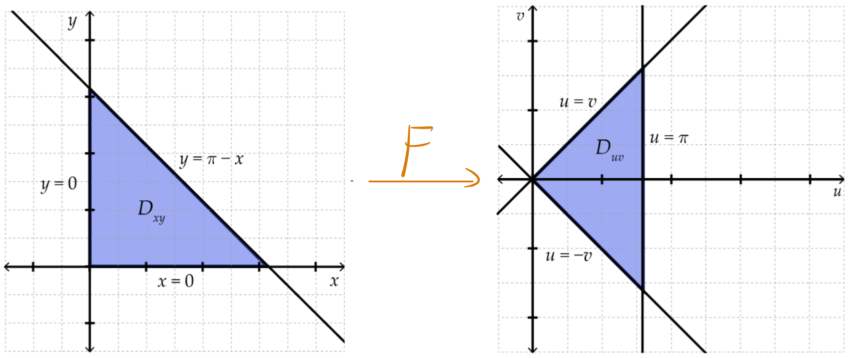
$V = \int_0^1 \int_0^1 5(4u + 3v)\, du\, dv = \frac{35}{2}$



Example 2

Use the mapping $(u, v) = F(x, y) = (x + y, -x + y)$ to evaluate

$$\int_0^{\pi} \int_0^{\pi - y} (x + y)\cos(x - y)\,dx\,dy$$



bounds $x = 0, \quad 0 \le y \le \pi$ $\quad \xrightarrow{F} \quad$ $v = y = u$

$y = 0, \quad 0 \le x \le \pi$ $\qquad v = -x = -u$

$x = \pi - y, \quad 0 \le x \le \pi$ $\qquad v = u - 2x = \pi - 2x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix} \right| = 2. \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\int_0^{\pi} \int_0^{\pi - y} (x+y)\cos(x-y)\,dx\,dy = \int_0^{\pi} \int_{-u}^{u} u\cos(-v) \left|\tfrac{1}{2}\right| dv\,du$$

$$= \frac{1}{2} \int_0^{\pi} \Big[-u\sin(-v)\Big]_{-u}^{u} du$$

$$= \pi$$

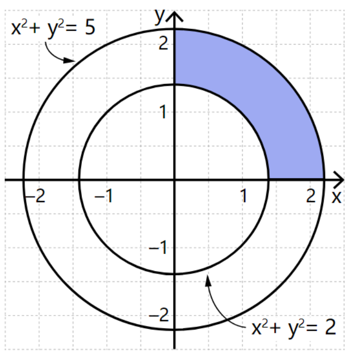
- Double integrals in Polar Coordinates

$$(x,y) = F(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Example 3

Evaluate $\displaystyle\iint_{D_{xy}} (2x+y)\,dA$ where D_{xy} is the quarter annulus shown below.



$x = r\cos\theta \qquad y = r\sin\theta$

$2x+y = 2r\cos\theta + r\sin\theta = r(2\cos\theta + \sin\theta)$

by change of variable theorem,

$$\iint_{D_{xy}} (2x+y)\,dA = \iint_{D_{r\theta}} r(2\cos\theta + \sin\theta)\,|r|\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_{\sqrt{2}}^{\sqrt{5}} r^2(2\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= 5^{\frac{3}{2}} - 2^{\frac{3}{2}}$$

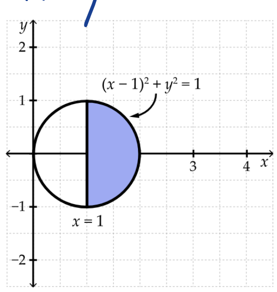
Example 4

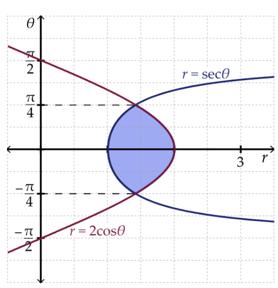
Evaluate $\displaystyle\iint_{D_{xy}} \frac{x}{x^2+y^2}\,dA$ where D_{xy} is the half disc $(x-1)^2+y^2 \le 1$, $x \ge 1$.

$x = r\cos\theta$

$x \ge 1 \Rightarrow r\cos\theta = 1 \qquad r = \sec\theta$

$x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos\theta \qquad r = 2\cos\theta$





$\sec\theta \le r \le 2\cos\theta \qquad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$$\iint_{D_{r\theta}} \cos\theta\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{\sec\theta}^{2\cos\theta} \cos\theta\,dr\,d\theta = 1.$$

When solving double integrals, the goal is always to write the integral as an iterated integral. This is done by converting the region D.

If D does not allow us to write an iterated integral, we can try to:

- change variables using the Change of Variables Theorem
- split D into a disjoint union using the Decomposition Theorem

If D allows us to write an iterated integral, but we are still getting stuck, we can try to:

- change the order of integration
- rewrite D to set up a different iterated integral
- perform a change of variables to set up a different iterated integral

14.1 Definition of triple integral.

DEFINITION
Integrable

A function $f(x, y, z)$ which is bounded on a closed bounded set $D \subset \mathbb{R}^3$ is said to be **integrable** on D if and only if all Riemann sums approach the same value as $\Delta P \to 0$.

DEFINITION
Triple Integral

If $f(x, y, z)$ is integrable on a closed bounded set D, then we define the **triple integral** of f over D, as

$$\iiint_D f(x, y, z)\, dV = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

DEFINITION
Average Value

Let $D \subset \mathbb{R}^3$ be closed and bounded with volume $V(D) \neq 0$, and let $f(x, y, z)$ be a bounded and integrable function on D. The **average value** of f over D is defined by

$$f_{avg} = \frac{1}{V(D)} \iiint_D f(x, y, z)\, dV$$

intepretation of triple integral: $\iiint_D f(x,y,z)\,dV$.

triple integral as volume : $V(D) = \iiint_D 1\,dV$. $f(x,y,z)=1 \quad (x,y,z)\in D$

triple integral as mass : $M = \iiint_D f(x,y,z)\,dV$

$\Delta M_i \approx f(x_i, y_i, z_i)\,\Delta V_i$

$M = \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i$

– properties

THEOREM 1

(Linearity)

If $D \subset \mathbb{R}^3$ is a closed and bounded set, c is a constant, and f and g are two integrable functions on D, then

$$\iiint_D (f + g)\, dV = \iiint_D f\, dV + \iiint_D g\, dV$$

$$\iiint_D cf\, dV = c \iiint_D f\, dV$$

THEOREM 2 **(Basic Inequality)**

If $D \subset \mathbb{R}^3$ is a closed and bounded set and f and g are two integrable functions on D such that $f(x,y,z) \leq g(x,y,z)$ for all $(x,y,z) \in D$, then

$$\iiint\limits_D f\,dV \leq \iiint\limits_D g\,dV$$

THEOREM 3 **(Absolute Value Inequality)**

If $D \subset \mathbb{R}^3$ is a closed and bounded set and f is an integrable function on D, then

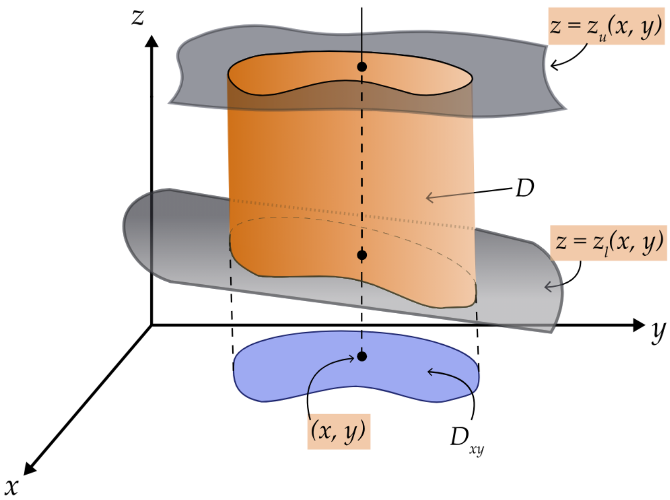
$$\left|\iiint\limits_D f\,dV\right| \leq \iiint\limits_D |f|\,dV$$

THEOREM 4 **(Decomposition)**

Assume $D \subset \mathbb{R}^3$ is a closed and bounded set and f is an integrable function on D. If D is decomposed into two closed and bounded subsets D_1 and D_2 by a piecewise smooth surface C, then

$$\iiint\limits_D f\,dV = \iiint_{D_1} f\,dV + \iiint_{D_2} f\,dV$$

14.2 Iterated integrals



THEOREM 1 Let D be the subset of $\mathbb{R}^3$ defined by

$$z_\ell(x,y) \le z \le z_u(x,y) \quad \text{and} \quad (x,y) \in D_{xy}$$

where z_ℓ and z_u are continuous functions on D_{xy}, and D_{xy} is a closed bounded subset in $\mathbb{R}^2$, whose boundary is a piecewise smooth closed curve. If $f(x,y,z)$ is continuous on D, then

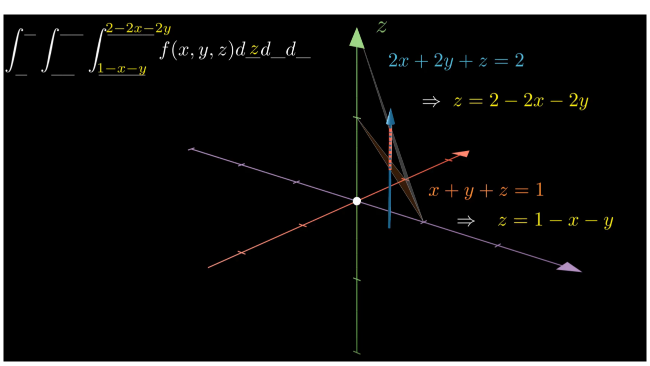
$$\iiint_D f(x,y,z)\,dV = \iint_{D_{xy}} \int_{z_\ell(x,y)}^{z_u(x,y)} f(x,y,z)\,dz\,dA$$

THEOREM 2 **(Basic Inequality)**

If $D \subset \mathbb{R}^3$ is a closed and bounded set and f and g are two integrable functions on D such that $f(x,y,z) \le g(x,y,z)$ for all $(x,y,z) \in D$, then

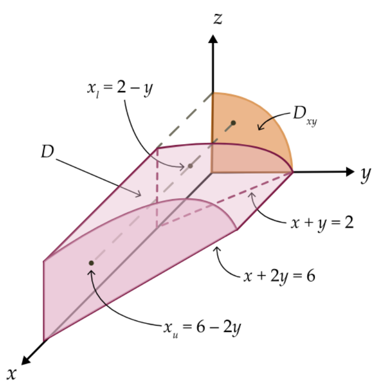
$$\iiint_D f\,dV \le \iiint_D g\,dV$$

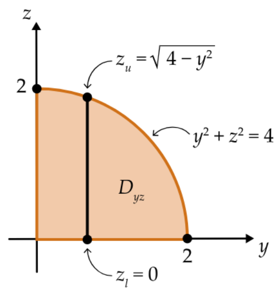
ep.



Example 2

Evaluate $\iiint\limits_D \frac{z}{4-y}\,dV$, where D is the region bounded by the cylinder $y^2+z^2=4$, and the planes $x+y=2$, $x+2y=6$, $z=0$, $y=0$, and lying in the first octant.





$$\iint\limits_{D_{yz}} \int_{2-y}^{6-2y} \frac{z}{4-y}\,dx\,dA$$

(也可以写作 $\int_0^2\int_{2y}^{6-2y}\int_0^{\sqrt{4-y^2}} \frac{z}{4-y}\,dz\,dx\,dy$)

$$= \iint\limits_{D_{yz}} z\,dA$$

$$= \int_0^2\int_0^{\sqrt{4-y^2}} z\,dz\,dy$$

$$= \frac{1}{2}\int_0^2 (4-y^2)\,dy$$

$$= \frac{8}{3}$$

14.3 Change of Variable Theorem

$$\iiint_{D_{xyz}} f(x,y,z)\, dV$$

THEOREM 1 **(Change of Variables Theorem)**

Let

$$x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w)$$

be a one-to-one mapping of D_{uvw} onto D_{xyz}, with g, h, k having continuous partials, and

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \neq 0 \quad \text{on} \quad D_{uvw}$$

If $f(x, y, z)$ is continuous on D_{xyz}, then

$$\iiint_{D_{xyz}} f(x, y, z)\, dV = \iiint_{D_{uvw}} f\big(g(u, v, w), h(u, v, w), k(u, v, w)\big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV$$

Example 1

Evaluate $I = \iiint_{D_{xyz}} x^2\, dV$, where D_{xyz} is the subset of $\mathbb{R}^3$ bounded by the surfaces $xy = 1$, $xy = 3$, and the planes $y + z = -1$, $y + z = 0$, $x + y + z = 1$ and $x + y + z = 2$.

→ level surface xy. $y+z$. $x+y+z$

D_{xyz} : $1 \le xy \le 3$ $-1 \le y+z \le 0$ $1 \le x+y+z \le 2$

→ define a mapping : $u = xy$. $v = y+z$ $w = x+y+z$

Jacobian : $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \det \begin{vmatrix} y & x & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = x$

By change of variable theorem, $I = \iiint_{D_{xyz}} x^2\, dx\, dy\, dz = \iiint_{D_{uvw}} x^2 \left| \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$

By inverse property of Jacobian, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \left[\dfrac{\partial(u,v,w)}{\partial(x,y,z)} \right]^{-1} = \dfrac{1}{x}$

$I = \iiint_{D_{uvw}} x\, du\, dv\, dw.$

$= \iiint_{D_{uvw}} (w - v)\, du\, dv\, dw$

$= \int_1^2 dw \int_{-1}^0 dv \int_1^3 (w - v)\, du.$

Cylindrical Coordinates

$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho\cos\varphi$

$\rho \geq 0 \qquad 0 \leq \varphi \leq \pi \qquad 0 \leq \theta < 2\pi$

Jacobian : $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \rho^2 \sin\varphi$.

proof:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det\begin{bmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix} = \rho^2\sin\varphi.$$

Example 1

Evaluate $I = \iiint\limits_{D_{xyz}} x^2\, dV$, where D_{xyz} is the subset of $\mathbb{R}^3$ bounded by the surfaces $xy = 1$, $xy = 3$, and the planes $y + z = -1$, $y + z = 0$, $x + y + z = 1$ and $x + y + z = 2$.

Step 1 : make change of variables $(x, y, z) = (au, bv, cw)$

derivative matrix : $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad \det A = abc$.

equation of ellipsoid : $u^2 + v^2 + w^2 \leq 1$.

Step 2 : change to spherical coordinates in ρ, φ, θ.

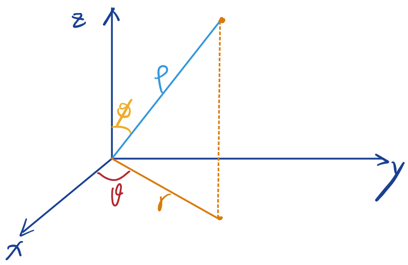
Jacobian : $\rho^2 \sin\varphi$.

equation of ellipsoid : $\rho^2 \leq 1$

Step 3 : evaluate the integral

$$V = \int_0^1 \int_0^{2\pi} \int_0^{\pi} abc\, \rho^2 \sin\varphi \, d\varphi\, d\theta\, d\rho = \frac{4}{3}\pi abc.$$

Spherical Coordinates



ρ : dist to origin

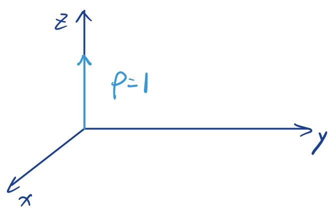
θ : usual polar angle $(0 \le \theta \le 2\pi)$

ϕ : angle with the z-axis $(0 \le \phi \le \pi)$

Example

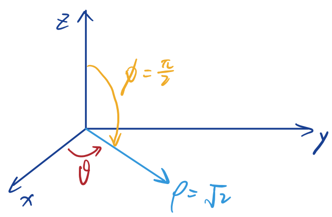
① $(x, y, z) = (0, 0, 1)$

$(\rho, \theta, \phi) = (1, 0, 0)$



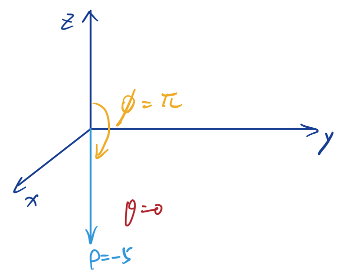
② $(x, y, z) = (1, 1, 0)$

$(\rho, \phi, \theta) = (\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4})$



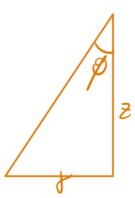
① $(x, y, z) = (0, 0, -5)$

$(\rho, \phi, \theta) = (5, \pi, 0)$



Formulas

$\cos\phi = \frac{z}{\rho}$ $\quad z = \rho\cos\phi.$ $\quad \sin\phi = \frac{r}{\rho}$ $\quad r = \rho\sin\phi$



So

$x = r\cos\theta = \rho\sin\phi\cos\theta.$

$y = r\sin\theta = \rho\sin\phi\sin\theta.$

$z = \rho\cos\phi.$ $\quad \rho = \sqrt{x^2+y^2+z^2}$ (相当于三维勾股定理)